This

\# 7345.

EUCLID.

Taken from a Brass Coin in the Repository of the late Queen Christian of Swed

This is an excellent copy of Euclid

Euclid *the Mathematician was of* Alexandria *where he Taught in the Reign of* Ptolomy Lagus *in the* CXX Olympiad *and Year of* Rome 454. *He Wrote many things relating to Musick and Geometry: But his XV Books of Element (of which he is generaly thought to be only the Collector) are most applauded, the two last are attributed to* Hypsicles *of* Alexandria *and not to him.* Cardan Vossius

THE ELEMENTS of EUCLID:

WITH

SELECT THEOREMS

OUT OF

ARCHIMEDES.

By the Learned ANDREW TACQUET.

To which are added,

PRACTICAL COROLLARIES, ſhewing the Uses of many of the Propoſitions.

1667-1752

By WILLIAM WHISTON,* M. A.

Mr. LUCAS's Profeſſor of the MATHEMATICKS in the UNIVERSITY of CAMBRIDGE.

In this EIGHTH EDITION is added an APPENDIX of PRACTICAL GEOMETRY, with Forty New Figures, and a Brief and Independent Demonſtration of certain Select and moſt uſeful Propoſitions.

By S. F.

DUBLIN:

Printed by and for I. JACKSON in Meath-ſtreet, 1753.

* Succeeds Sir Isaac

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AN

HISTORICAL ACCOUNT OF THE RISE and PROGRESS OF THE MATHEMATICKS.

IT ſeem'd meet to me, when I was about to ſet forth the Elements of the *Mathematicks*, to premiſe a few Things concerning the Riſe and Excellency of this Science, that its Candidates may underſtand what a Kind of Science it is to which they are about to dedicate themſelves; and that it may be made manifeſt againſt thoſe, who ſlight thoſe Things, whereof they are ignorant of how great Value and Dignity this Knowledge is, which the wiſeſt Men of all Ages have, with incredible Study, labour'd to attain unto, and become poſſeſs'd of. Moreover, I muſt own that *Peter Ramus*'s Labours have been of great Service to me in the com-

piling of this Account, who in the whole First Book of his Institution, which is not a little one, hath out of *Proclus*, *Laertius*, *Gellius*, *Polybius*, *Tzetzes*, and others, composed a Mathematical History both accurately and copiously.

The Mathematical Sciences were the first of all other amongst Men, if we may believe *Josephus*. He, *Book* I. Chap. 3. writeth, That the Posterity of *Seth* observed the Order of the Heavens, and the Courses of the Stars. And lest these Inventions should slip out of the Knowledge of Men, *Adam* having predicted a two-fold Destruction of the Earth, one by a Deluge, the other by Fire, they raised two Columns, one of Bricks, of Stone the other; and inscribed their Inventions upon them, that if the Brick one should happen to be destroy'd by the Deluge, that of Stone, which would remain, might afford Men an Opportunity of being instructed, and present to their View the Things which it had inscribed on it. They say also, that that Stone Pillar, which even in our Days is seen in *Syria*, was dedicated by them. This *Josephus* says; whom I leave to vouch for the Story.

That, the *Assyrians* and *Chaldeans* were the first of Mortals, after the Flood, who applied themselves to the *Mathematicks*, is delivered by the same *Josephus*; as also by *Pliny*, *Diodorus* and *Cicero*. But the *Mathematick Arts* which first sprang among the *Chaldeans*, amongst whom they flourished, were afterwards transferr'd out of *Chaldea* and *Assyria* unto the *Egyptians*, by *Abraham*. For, when, at the Command of God, he went forth from his native Soil into

Palestine,

Palestine, and from thence into *Egypt*, and perceiv'd the *Egyptians* to be taken with the Study of good Arts, and to be of a very notable Wit and Capacity for Learning, (as *Josephus* testifies, *Book* I. Chap. 9.) he communicated to them Arithmetick and Astronomy; and consequently Geometry, which must of Necessity go before Astronomy. In which Studies afterwards the *Egyptians* so flourish'd, that *Aristotle*, 1 *Metaph.* Chap. 1. doth affirm, That the *Mathematick Arts* were first found out in *Egypt*, by their Priests; who, by their Employments, were at leisure for these Things.

Then these Arts crossing the Sea out of *Egypt*, came to the Philosophers of *Greece*: For *Thales* the *Milesian*, who flourish'd 584 Years before Christ, was the first of the *Greeks*, who coming into *Egypt*, transferr'd Geometry from thence into *Greece*. He it was indeed, who, besides other Things, found out the 5th, 15th and 26th Propositions of the First Book. To the same are also owing the 2d, 3d, 4th, 5th, of the Fourth Book. The same Person began to observe the Equinoxes and Solstices, as *Laertius* testifies; and he was the first who foretold an Eclipse of the Sun, as *Hippias* and *Aristotle* do write; and *Tzetzes* saith, That he also foreshew'd an Eclipse of the Moon to King *Cyrus*. For which Things sake he is to be look'd on as the first Founder and Author of the Mathematical Sciences in *Greece*.

After him was *Pythagoras* of *Samos*: Which most antient Philosopher, exceedingly improv'd and adorn'd the Mathematick Sciences. And he so gave himself to Arithmetick in particular, that

that almoſt his whole Method of Philoſophizing was taken from Numbers. And he firſt of all, as *Laertius* relates, abſtracted Geometry from Matter; in which Elevation of the Mind, he found out the 32d, 44th, 47th and 48th Propoſitions of the Firſt Book. But he is eſpecially celebrated for the Invention of *Prop.* 32, and 47. of that Book; and he conceiv'd ſo great Joy upon this Invention, that, as *Apollodorus* witneſſes in *Laertius*, on that account he ſacrificed an Hecatomb. The ſame Perſon firſt laid open the Matter of incommenſurable Magnitudes, and the Five regular Bodies. The ſame Perſon did both moſt diligently teach and exerciſe the Art of Aſtrology and Muſic: For he did not only acutely and ſubtilly find out many Things himſelf, but he alſo firſt opened a School, in which Youth might learn theſe honourable and noble Arts.

Pythagoras was follow'd by *Anaxagoras* of *Clazomenæ*, and *Oenopides* of *Chios*, of whom *Plato* makes mention in his Dialogue, *The Lovers*, where young Men are brought in contending about *Anaxagoras* and *Oenopides* in their Deſcriptions of Circles. *Ariſtotle* reports, that a certain Treatiſe of Geometry was written by *Anaxagoras*; and we have it from *Laertius*, that it was ſhew'd by him that the Sun is greater than *Peloponneſus* (a notable Inſtance of the Infancy of Aſtronomy at that Time) and that he made ſome Conjectures concerning Habitations in the Moon. As for *Oenopides*, to him *Proclus* aſcribes the 12th and 13th, *L.* 1. Theſe were followed by *Briſo*, *Antipho*, and *Hippocrates* of *Chios*, all of them, for attempting the

Quadrature

Quadrature of the Circle, reprehended by *Ariſtotle*, and at the ſame Time celebrated. But amongſt them, *Hippocrates* was by far the moſt Famous; that celebrated Perſon, who, of a Merchant, growing to be a Philoſopher, and a Geometrician, beſides the Quadrature of the Circle, alſo firſt attempted the Doubling of the Cube, by two mean Proportionals, which, as being an excellent, and indeed the only Way, all that have followed him to this Time have embraced it. 'Tis alſo his peculiar and great Commendation, that he, as *Proclus* teſtifies, firſt wrote Elements, and digeſted into Order the Diſcoveries made by others.

Democritus was admirable, not in Philoſophy only, but alſo in the Mathematicks. His Phyſical Monuments, and, if ſuch there were, his Mathematical Works alſo, are wholly loſt, through the Envy (as ſome report) of *Ariſtotle*, who deſired to have no other Writings read, but his own. The Philoſophy of *Democritus* hath been reſtored by *Peter Gaſſendus*, in a moſt Learned Work lately put forth. *Theodorus Cyrenæus*, although none of his Mathematical Inventions are extant, yet is great upon this account, if there were no other, that he is reported to have been the Maſter of *Plato*.

Unto *Plato* therefore we are come at length, than whom no one brought greater Luſtre to the Mathematical Sciences. He amplified Geometry with great and notable Additions, beſtowing incredible Study upon it. And above all, the Art Analytic, or of Reſolution, was found out by him, the moſt certain Way of In-

vention and Reaſoning. He ſet off and illuſtrated his Books of Philoſophy in a Mathematical Way, and encourag'd whatſoever was admirable in Mathematical Philoſophy. Upon the Door of his Academy was read this Inſcription: οὐδεὶς ἀγεωμέτρητος εἰσίτω: *Let no one ignorant of Geometry enter here*; an illuſtrious Inſtance to demonſtrate, how the Mathematicks are not foreign, but proper, not unuſeful, or unbecoming, but honourable and profitable to ſound and certain Philoſophy. In a Word, how great both Admirer and Maſter of the Mathematicks *Plato* was, that Man will of himſelf eaſily underſtand, who ſhall read his Monuments through.

Out of *Plato*'s Academy, almoſt innumerable Mathematicians came forth. Thirteen of *Plato*'s familiar Acquaintance are commemorated by *Proclus*, as Men by whoſe Studies the *Mathematicks* were improv'd. From hence were *Leodamus* the *Thaſian*, *Archytas* the *Tarentine*, *Theætetus* the *Athenian*, by whom the *Mathematicks* were notably enlarged. *Leodamus* practiſed the Analyſis received from *Plato*, and is ſaid by *Laertius* to have found out many Things by the Help of it. As for *Theætatus*, both to his own Inventions, amongſt which are the Elements written by him, and the Inſcription of regular Bodies; and *Plato*'s Encomiums, who alſo inſcribed a Dialogue to his Name, do make him famous.

Archytas alſo wrote Elements himſelf; and his Doubling of the Cube is mentioned by *Eutocius*; whoſe ſingular Commendation it likewiſe was, that he was almoſt the Firſt that

brought

brought down the *Mathematicks* to Human Uſes; by whoſe Contrivance alſo a wooden Pidgeon was made to fly, as *Gellius* reports; he being preceded by *Dædalus*, and followed by other Artificers, yielded Matter for the Fables of the Poets. Moreover, *Archytas* was both a Mathematician and General of an Army: He five times commanded the Forces of his own Citizens, in the Wars of his Country, and five times overcame their Enemies. The meer Name of *Neoclides* is only Famous, he being more illuſtrious for his Scholar *Leon* perhaps, than for his own Inventions. *Leon* certainly wrote Elements of all the *Mathematicks*, improv'd them, and made them more fit for Uſe. Wherefore he is deſervedly to be reckon'd amongſt the chief Compilers of Elements.

Eudoxus of *Cnidos* was not inferior to *Leon*: A Man great in Arithmetick, and to him, (if we may believe the *Greek* Scholiaſt) we owe the whole Fifth Book. He likewiſe wrote Elements, and made them more general, and encreaſed the Sections begun by *Plato*; over and above this he was the firſt Framer of Aſtronomical Hypotheſes, and derived down the Springs of Geometry, as *Archytas* had done before, to Mechanicks. *Amyclas* the *Heracleot*, and *Menæchmus*, and his Brother *Dinoſtratus*, *Helicon* of *Cyzium*, *Theudius*, *Hermotimus* the *Colophonian*, *Philippus* the *Medmæan*, all Platoniſts rendered Geometry much more perfect. *Menæchmus* alſo found out the *Conic* Sections, and by the help of them, two mean Proportionals; whoſe Invention in this Caſe is preferr'd by *Eutocius* before

any other. *Theudius* and *Hermotimus* made the Elements more univerſal and full. And all theſe, who were of *Plato*'s Academy, brought Mathematick Philoſophy to Perfection, as *Proclus* ſaith. *Xenocrates* alſo, one of *Plato*'s Auditors, and Maſter of *Ariſtotle*, as well as *Ariſtotle* himſelf, were famous for the Knowledge of the Mathematicks. When a certain Perſon, who knew nothing of Geometry, was minded to be his Auditor, *Go thy Way*, ſaith he, *for thou wanteſt the very Handles of Philoſophy.*

But of *Ariſtotle*, what can I ſay? All his Books are filled with Mathematical Arguments, out of a Collection of which *Blancane* hath made a Book. Two of *Ariſtotle*'s School are eſpecially celebrated, *Eudemus* and *Theophraſtus*: This latter wrote two Books of Numbers, four of Geometry, and one of indiviſible Lines: The other, compoſed a Mathematical Hiſtory; and from him *Proclus*, and others have borrowed theirs. To *Ariſteus*, *Iſidore*, *Hypſicles*, moſt ſubtil Geometricians, we are eſpecially indebted for the Books of Solids. Laſtly, *Euclid* gathered together the Inventions of others, diſpoſed them into Order, improv'd them, and demonſtrated them more accurately, and left to us thoſe *Elements*, by which Youth is every where inſtructed in the Mathematicks. He died in the Year before Chriſt, 284 There followed *Euclid* almoſt an 100 Years afterwards *Eratoſthenes* and *Archimedes*. The Name of *Eratoſthenes* was very famous, but his Writings are loſt. Many Remains we have of *Archimedes*, and many we have loſt.

But

But when I name *Archimedes*, I conceive in my Mind the very Top of Human Subtilty, and the Perfection of the whole Mathematical Sciences. His wonderful Inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetzes* and others. *Conon* was Contemporary to *Archimedes*, one who was both a Geometrician and an Aſtronomer, whoſe Death *Archimedes* laments in his Book of the Quadrature of the *Parabola*. There followed *Archimedes* and *Conon*, and that at no great Diſtance, *Apollonius* of *Perga*, another Prince in Geometry, who was called by way of high Encomium, *The Great Geometrician*. There are extant Four [now Seven] moſt ſubtil Books of his *Conic's*. To the ſame Perſon are aſcribed the Fourteenth and Fifteenth Books of *Euclid*, which were contracted by *Hypſicles*. *Hipparchus* and *Menelaus* wrote, this latter, Six, the other, Twelve Books of Subtenſes in a Circle; for which Invention, ſo very profitable and neceſſary, great Commendations and Thanks are due to both. There are alſo extant three Books of *Menelaus* concerning Spherical Triangles. Three moſt uſeful Books of Spheric's of *Theodoſius* the *Tripolite* are alſo in the Hands of all. And theſe indeed, if you except *Menelaus*, lived all of them before Chriſt.

In the Year after Chriſt, 70. there appeared in the World, *Claudius Ptolomæus*, the Prince of Aſtronomers, a Man certainly wonderful, and (as *Pliny* ſaith) above the Nature of Mortals. But he was not only moſt skilful in Aſtronomy, but in Geometry alſo; which, as many other Things

written by him do witneſs, ſo eſpecially do the Books of Subtenſes: Thoſe of *Menelaus*, which were Six, and the Twelve of *Hipparchus*, all contracted by him into five Theorems. As for *Plutarch*, a moſt fam'd Philoſopher, there are extant his Mathematical Problems. And all know of the learned Commentaries of *Eutocius* the *Aſcalonite* upon *Archimedes*. By him are recited the Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sphorus*, *Heron*, as of ſo many excellent Maſters in the Mathematicks, concerning Doubling the *Cube*. *Heron*'s Genius certainly was excellent, as well for Mechanicks as Geometry. The Doubling the *Cube* delivered by him, is commended by *Pappus*, Book III. *Prop.* 7. before all other. The admirable Works of *Cteſibius* the *Alexandrian*, to whom we owe our Pumps, are celebrated by *Vitruvius*, *Proclus*, *Pliny* and *Athenæus*. The Name alſo of *Geminus* is not in the loweſt Place amongſt Mathematicians, whom *Proclus* has preferr'd in many Things before *Euclid* himſelf.

Diophantus, and he alſo an *Alexandrian*, was as great in Arithmetick, as *Archimedes*, *Apollonius* or *Euclid* in *Geometry*; he was certainly a Maſter of all Subtilty relating to Numbers: By him was found out that admirable Art, which we call *Algebra*; which in theſe Times has been rendered more perfect and univerſal by *Francis Vieta*, and *Renatus Carteſius*. There are others who are celebrated amongſt the Antients alſo; as *Nichomachus*, famous for Arithmetical, Geometrical and Muſical Monuments; *Serenus* well known to Geometricians for his two Books,

concerning

concerning the Section of a Cylinder; *Proclus*, *Pappus*, *Theon*. How great a Mathematician *Proclus* was, is manifeſt from his learned Commentaries on *Euclid*, and other Writings. And this is he, I ſuppoſe, who, as *Zonarus* reports, and from him *Ramus* and *Baronius*, about the Year of Chriſt 514. with Optic Artifice, and the Glaſſes which he uſed, burnt the Fleet of *Vitalian*, who was beſieging *Conſtantinople*. The Praiſes of *Theon*, which truly are deſervedly great, *Peter Ramus* wonderfully exaggerates; inſomuch, that even the Books which hitherto all have aſcribed to *Euclid*, ought, as he thinks, to be attributed to *Theon*. But *Ramus*, who every where is ready to detract from *Euclid*, and this without grounding himſelf upon any ſolid Foundation, is not to be hearken'd to here. To come at length to a Concluſion, let *Pappus* bring up the Rear, the laſt in Time amongſt the Antients, as being one who liv'd about the Year 400; but in Reputation, and all Mathematical Commendation, to be reckon'd amongſt the firſt. *Alexandria*, that City ſo fruitful of great Men, which before had brought forth *Hypſicles*, *Cteſibius* and *Diophantus*, produced him alſo, to the great Advantage of the Mathematicks. He wrote Seven Books of Mathematical Collections, of which the Two firſt are loſt. The Five other do abound with ſo many, and ſuch various moſt noble Inventions in almoſt all Parts of the Mathematicks, that they are eſteemed amongſt the chief Monuments of the Antients which are extant.

And thus you have a ſhort Hiſtory of the Origin and Progreſs of the Mathematicks. From which appears the Antiquity, Excellency and Dignity of this Science. Certainly the ſame eminent Perſons in the Commonwealth of Learning, who diſcover'd Philoſophy, diſcover'd alſo the Mathematicks, like two Siſters born at one Birth ; whom, if any one would violently ſeparate from each other, he certainly attempts to break off their Native Concord, with moſt notable Injury, and as it were Cruelty to both; ſeeing, as it is wont to fall out in the Caſe of Twins, where they are remov'd from one another, in Place or by Death, ſo it will be like to happen here, that Mathematicks being plucked away from her, Philoſophy muſt needs languiſh and pine away.

N. B. *Q. E. D.* or *W. W. D.* is which was to be demonſtrated.
Q. E. F. which was to be done.
Q. E. I. which was to be found.

Doctor *Barrow's* Words prefix'd before his *Apollonius.*

GOD always acts Geometrically.

HOW great a Geometrician art thou, O Lord! For while this Science has no Bound; while there is for Ever room for the Discovery of New Theorems, even by Human Faculties, Thou art acquainted with them all at one View, without any Train of Consequences, without any wearisom Application of Demonstration. In other Arts and Sciences our Understanding is able to do almost nothing; and, like the Imagination of Brutes, seems only to dream of some uncertain Propositions: Whence it is, that in so many Men are almost so many Minds. But in these Geometrical Theorems all Men are agreed: In these the Human Faculties appear to have some real Abilities, and those Great, Wonderful and amazing. For those Faculties which seem of almost no Force in other Matters, in this Science appear to be Efficacious, Powerful and Successful, &c. Thee therefore do I take hence occasion to Love, and Rejoice in, and Admire; and to pant after that Day, with the Earnest Breathings of my Soul, when thou shalt be pleased,

out of thy Bounty, out of thy Immense and Sacred Benignity, to grant me the Favour to perceive, and that with a pure Mind, and clear Vision, not only these Truths, but those also, which are more numerous, and more important; and all this without that continual and painful Application of the Imagination, which we discover these withal, &c.

Mathematical Notes, or Abbreviations.

$=$ The Note for Equality. So $a = b$ signifies that a and b are equal.

$+$ The Note for Addition. So $a + b$ signifies the Sum of a and b together.

$-$ The Note for Subtraction. So $a - b$ signifies the Difference between a and b

$\times$ The Note for Multiplication. So $a \times b$, or $a\,b$ signifies a multiplied by b.

$::$ The Note for Equality of Proportion. So $A : B :: a : b$ signifies that A bears the same Proportion to B, that a bears to b.

$\div$ The Note of continued Proportion. So $A\,B\,C \div$ signifies that A bears the same Proportion to B, that B bears to C.

q The Note for a Square. So $C\,B\,q$ signifies the Square of the Line $C\,B$.

c The Note for a Cube. So $C\,B\,c$ signifies the Cube of of the Line $C\,B$.

THE

THE
Elements of EUCLID.

BOOK I.

DEFINITIONS.

A Point is a Mark in Magnitude, which is [ſuppoſed to be] indiviſible.

That is, which cannot be divided ſo much as in Thought. A Point is the Beginning, as it were, of all Magnitude, as Unity is of Number.

2. A Line is a Magnitude which hath Length only, and wants all Breadth; foraſmuch as it is underſtood to be produced from the flowing of a Point.

3. Points are the Terms of a Line. *Fig.* 1.

4. A right Line, is that which lies evenly betwixt its Terms. *Table* 1.

Or, as *Archimedes*; a right Line is the leaſt of all thoſe which have the ſame Terms; or, is the ſhorteſt of all thoſe which can be drawn betwixt two Points.

Or, as *Plato* hath it; a right Line is that whoſe Extremes hide all the reſt: [That is, when the Eye is placed in a Continuation of the Line.]

The Senſe is the ſame in all. The Inſtrument whereby right Lines are deſcribed, is [called] a Rule; which, whether it be ſtrait or not, you may know by this Tryal.

Deſcribe a Line according to the Rule; then turning the Rule ſo that, that which before was the Right-hand End may now become the Left-hand End, apply it again

to the Line before described; if it doth now entirely fall in with the Line, the Rule is ſtrait, if not, the Rule is not ſtrait. The Reaſon hereof depends on *Axiom* 13.

5. A Surface is a Magnitude which hath only Length and Breadth.

It hath two Dimenſions therefore: And is underſtood to be produced by the flowing of a Line.

6. Lines are the Extremes of a Surface.

7. A Plane, or a plain Surface, is that which lies evenly betwixt its extreme Lines.

Or as *Hero*, that, to all the Parts whereof a right Line may be accommodated.

For it is produced from the Motion of a right Line.

Or, a plain Surface is that whoſe Extremes any of them hide all the reſt, [the Eye being placed in a continuation of the Surface.]

Or, it is the leaſt of all Surfaces which have the ſame Terms. The Senſe is the ſame in all.

Euclid hath not here defined a Body or Solid, becauſe he was not yet about to treat concerning it. But leſt any one ſhould want the Definition thereof, take it here thus; A Body is a Magnitude, long, broad and deep. A Body therefore hath three Dimenſions, a Surface two, a Line one, a Point none.

8. A plain Angle is the mutual Inclination to each other of two Lines, which touch one another in a Plane, and ſo as not to make one Line.

Fig. 2, 4. Therefore the two Lines A B, C A touching one another in A, but ſo as not to make one Line, conſtitute an Angle.

9. The Sides or Legs of an Angle are the Lines which make the Angle.

10. The *Vertex* or Top of an Angle is the Point (A) in which the Legs do meet and touch one another.

Note, That a ſingle Angle is deſigned by one Letter put at the Top: When there are more at one Point, they are deſigned by three Letters, the middlemoſt of which denotes the Top of the Angle; and many Times alſo by one Letter interpos'd betwixt the Sides near the Top. So in *Fig.* 5. the Angle made by the Lines B A, C A is deſigned either by three Letters B A C, or by one only O.

11. Angles are called Equal, if when the Tops of them are laid upon one another, the Sides of one agree with the Sides of the other. But unto this it is not required that the Sides ſhould be of an equal Length. 12. They

12. They are called Unequal, when the Top and one Side agreeing, the other doth not agree; and that is called the Greater, whose Side falls without. So the Angle BAE is greater than the Angle B A C. *Fig.* 5.

An Angle is not diminish'd or increas'd by the Diminution or Augmentation of the Sides that include it.

13. A right-lin'd Angle is that which right Lines constitute; a curvi-lineal, which crooked Lines; a mixt one, that which a right Line and a crooked one make. *Fig.* 2, 4.

14. When the right Line [C A] standing upon the right one [B F] leans unto neither Part, and therefore makes the Angles on both Sides equal, C A B=CAF both of the equal Angles are called Right ones: But the right Line C A which stands upon the other, is called a perpendicular Line, or barely a Perpendicular. *Fig.* 6.

A right Angle may also be defined thus. *Fig.* 6.

A right Angle is that, that (B A C) when on the other Side an equal one ariseth (C A F) if you produce or draw forth a Side, as (B A.)

Two Rules so joined as to contain a right Angle, make an Instrument, which is called a Square. *Pythagoras* was the Inventor of it, as *Vitruvius* affirmeth *c.* 2. *l.* 9. So great is the Use and Force of a right Angle in Framing, Measuring and Strengthning all Things, that nothing almost can be done without it. The Proof of a Square is made thus: Apply the Side of it, AE to the right Line A F, and describe the right Line C A along the other Side. Then turning the Square towards B, if on both Sides it agrees to the right Lines C A, A B, you may know that it is true and exact. The Reason hereof appears from the 14th *Definition* itself.

15. The Angle B A C, which is greater than the right one F A C, is called an obtuse Angle. *Fig.* 7.

16. The Angle (LAC) which is less than the right Angle (FAC) is called an Acute one. *Fig.* 8.

17. A plain Figure is a plain Surface, bounded on every Side with one or more Lines.

18. A Circle is a plain Surface, contained within the Compass of one Line, called the Circumference; from which Line all the right Lines that can be drawn unto one certain Point, within the contained Space (A) are equal. *Fig.* 9.

19. That Point is called the Center.

20. The Diameter is a right Line (B A) drawn thro' the Center, and on both sides ended at the Circumference; *Fig.* 9.

and consequently it divides the Circle into two equal Parts. (As is abundantly manifest from the exact Agreement of two Semi-circles when laid upon one another.)

21. The Semi-diameter, or *Radius*, is the right Line A F, drawn from the Center to the Circumference.

22. The Semi-circle is a Figure (B L C) which is contained by the Diameter B C, and half the Circumference (B L C)

Mathematicians are wont to divide the Circumference into 360 equal Parts (which they call Degrees) the Semi-circumference into 180, the Quadrant, or Quarter, into 90.

23. A Right-lin'd Figure, is a plain Surface bounded on every side with right Lines.

Fig. 10. 24. A Triangle is a plain Surface contained by Three right Lines.

This is the first and most simple of all Right-lin'd Figures, and that into which they are all resolved,

Fig. 10. 25. An Equilateral Triangle, is that which hath all the sides equal.

Fig. 11, 12. 26. An *Isosceles*, or Equicrural Triangle, is that which hath only two Sides equal.

Fig. 13. 27. A *Scalenum*, is that which hath Three unequal Sides.

Fig. 13. 28. A Right-angled Triangle, is that which hath one Angle right.

Fig. 12. 29. An Obtuse-angled Triangle, is that which hath One obtuse Angle.

Fig. 10, 11. 30. An Acute-angled Triangle, is that which hath Three acute Angles.

Fig. 14, 15. 31. Amongst Quadrilateral Figures, the Rectangle is that which hath four right, and consequently equal Angles; whether the sides be equal or not.

Fig. 15. 32. A Square, is that which hath equal sides, and is Right-angled, and consequently Equi-angled.

Every Square is a Rectangle; but every Rectangle is not a square.

Fig. 16. 33. A Rhombus is a Quadrilateral, or four-sided Figure, which is Equilateral, but not Equi-angled.

Fig. 17. 34. A Rhomboides, is that which hath the opposite sides and Angles equal, but is neither Equilateral, nor Equi-angled.

Fig. 14, 15, 16, 17. 35. A Parallelogram, is a Quadrilateral Figure, which hath each Two of its opposite sides (AB, F C and BF, AC) parallel

parallel to each other. Now what parallel Lines are, will be ſhewed in the following *Definition*.

Every Rectangle and Square is a Parallelogram; but every Parallelogram is not a Rectangle or a Square.

36. Right Lines are parallel, or equi-diſtant, which being in the ſame Plane, and drawn out on both Sides infinitely, are diſtant from one another by equal Intervals. *Fig.* 18.

The Intervals are ſaid to be equal, in reſpect of the Perpendiculars. Wherefore if all the Perpendiculars (Q L) unto one of the two Parallels (A B) ſhall be equal, the right Lines (A B, C F) are ſaid to be parallel.

Parallels are produced, if the right Line (L Q) which is perpendicular to the right Line (A B) be moved along (A B) always perpendicularly; for then its Extremities L deſcribes the Parallel C F.

37. The Diameter, or Diagonal of a Parallelogram, and every Quadrilateral, is a right Line (A F) drawn through the oppoſite Angles. *Fig.* 17.

38. Plain Figures contained by more Sides than Four, are called Many-ſided, or Many-angled, and by a *Greek* Word, *Polygons*.

39. The external Angle of a Right-lin'd Figure, is that which ariſeth without the Figure, when the Side is produced. Such are F B C, G C A, H A B. Every Figure therefore hath ſo many external Angles as it hath Sides, and internal Angles. *Fig.* 19.

Poſtulates.

A Poſtulate is that which is manifeſt in itſelf, that it may eaſily be done, or conceived to be done. It is required therefore to be granted that we may,

1. From any Point given draw a right Line unto any other Point given.

2. Draw forth a finite right Line in Length ſtill farther.

3. From any Center at any Interval deſcribe a Circle.

Axioms.

AN *Axiom* is a Truth manifeſt of itſelf.

1. Thoſe Things which are equal to the ſame thing, are equal alſo amongſt themſelves. And that which is greater or leſs than one of the Equals, is alſo greater or leſs than the other of them.

2. If to Equals you add Equals, the Wholes will be equal.

3. If from Equals you take away Equals, the Remainders will be equal.

4. If to Unequals you add Equals, the Wholes will be unequal.

5. If from Unequals you take away Equals, the Remainders will be unequal.

6. What Things are each of them half of the ſame Quantity, are equal amongſt themſelves; and what things are double or treble, or quadruple of the ſame, are equal amongſt themſelves.

7. What things do mutually agree with one another are equal.

8. If right Lines be equal, they will mutually agree with one another; and the ſame thing is true of Angles.

9. The Whole is greater than its Part.

10. All right Angles are equal amongſt themſelves.

11. Parallel Lines have a common Perpendicular: That is, the right Line which is perpendicular to one of them, is perpendicular alſo to the other.

Fig. 21 12. The two perpendicular Lines (L O, Q I) intercept equal Parts of the Parallels, L I, O Q.

13. Two right Lines do not comprehend a Space; for unto this there are required Three at the leaſt.

14. Two right Lines cannot have one common Segment; for that they cut one another only in a Point.

Of Propoſitions ſome propoſe ſomething to be done, and are called *Problems*; in others we proceed no further than bare Contemplation, which therefore are named *Theorems.*

PRO-

PROPOSITIONS.

THE requisite Citations are found in the Margin. When Propositions are cited, the first Number designs the Proposition; the Letter *l*, with the Number following, signifies the Book. As when you meet with (*per* 5. *l.* 3.) you must read it thus, (By the Fifth Proposition of the Third Book.) The Figure is always to be sought amongst the Figures of that Book in which we are then conversant. The rest of the Citations are easy to be understood.

The primary Affections of Triangles and Parallelograms are delivered in this Book. The more famous Propositions are, 32, 35, 37, 41, 44, 45, 47.

PROPOSITION I. Problem. *Fig.* 23.

UPON a given right Line (AB) to make an Equilateral Triangle.

From the Center A, with the Interval (A B) (*a*) describe the Circle FCB; and from the Center B with the same Interval B A describe the Circle ACL, cutting the former in the Point C, from which Point draw the right Lines C A, C B.

(a) *Per Postul.* 3.

I say, that the Triangle A C B now made, is Equilateral. For the right Line A C is equal to the right (*b*) Line A B, seeing they are Semi-diameters of the same Circle F C B. And again, the right Line B C is equal to the same right Line B A, seeing they are both Semi-diameters of the Circle L C A. Therefore, A C, B C are (*c*) equal betwixt themselves. And therefore all the Sides of the Triangle are equal. Therefore the Triangle (*d*) A C B is both Equilateral, and made upon the given Line A B; which was the thing to be done. *Q. E. F.*

(b) *Per Def.* 18.

(c) *Per Axiom* 1.

(d) *Per Def.* 25.

Fig. 77.

Corollary. " Hence we may measure an inaccessible " Line, as A B. For suppose any Equilateral Triangle " whatsoever B D E applied to the Point B along the Line " B A. Looking from the Point B, along the Line B E, " mark as many Points as you conveniently can in the Line " B C. Then remove the Triangle B D E along the Line " B C, from one place to another of that Line, until, by

taking

" taking aim along the ſide of the Triangle E D or C F, " you ſee the inacceſſible point A in a Continuation of that " Line. Thus the Triangle B A C is as well Equilateral " as B D E. If therefore you ſhall now meaſure the ac- " ceſſible Line B C, you have the meaſure of the inacceſſi- " ble A B. *Q. E. F.*

PROP. II. Problem.

Fig 24 *FROM a given Point A, to draw a right Line equal to one given, E F.*

Take with a pair of Compaſſes the Interval E F, and transfer it from A to D, the right Line A D will be equal to the given E F.

PROP. III. Problem.

Fig. 24. *TWO unequal right Lines being given, from the greater of them G H to cut off G I equal to the leſs E F.*

Take with a pair of Compaſſes the Interval of the leſs given Line E F, and transfer it unto the greater from G to I.

PROP. IV. Theorem.

Fig. 25. *IF in two Triangles (X, Z) one ſide of the one (B A) be equal to one ſide (F L) of the other, and another ſide (C A) of the one equal to another ſide (I L) of the other, and the Angles (A and L) made by theſe ſides be alſo equal; then the Baſes (B C, F I) are likewiſe equal, as alſo the Angles at the Baſes (B F and C I) which are oppoſite to equal ſides, and conſequently the whole Triangles are equal.*

For

For if we ſuppoſe the Triangle Z to be laid upon the Triangle X, the Sides L F, L I will perfectly agree and fall in together with the Sides that are equal to them, A B, A C, and this in ſuch ſort (*c*) that the three Points (L. F. I.) ſhall fall upon three Points, (A. B. C.) Therefore the whole Baſe F I will alſo fall upon the whole Baſe B C. But then the Angles F, B, and likewiſe thoſe, I, C, and the whole Triangles will mutually (*congruere*) agree to each other. All therefore, by the 7th *Axiom*, are equal. *Q. E. D.* Which was the Thing to be demonſtrated. (c) *Per Axio* 8.

Corollary. " (1.) Hence we may alſo in another way " meaſure the Line A B, although otherwiſe impracticable " by reaſon of ſome Obſtacle, as a River, *&c.* between " the Extremities thereof. For from any point whatſo- " ever, as the Point C, let the Angle A C B be obſerved, " and then let the Lines A C, B C be meaſured; and in " any acceſſible Plane let there be meaſured about the " Angle F, which is equal to the Angle C, two Lines F D " and F E, which are equal to the Lines A C and B C " reſpectively. And there will be the acceſſible Line D E " equal to the inacceſſible A B. *Q. E. I.* *Fig.* 78.

Corollary. (2.) " Hence alſo thoſe who play at Bil- " liards with Ivory Balls, may learn how by the Reflexion " of their own to hit and remove their Adverſary's Ball. " For let B be the Ball to be ſtricken, A that which is to " ſtrike it, and C D the Rectilinear Plane. Let the Line " B E be perpendicular to the Line C D, and D E be " equal to D B. If the Ball A be ſtricken and carried a- " long the right Side A F E unto the Point F, it will there " be ſo reflected, that after the Reflexion it will tend unto " B. For in the Triangles B F D, E F D, the Side F D " is common to both, and the Side B D is equal to the " Side D E; and the Angles at D are equal, as being " right ones. The whole Triangles therefore are equal; " and therefore the Angle B F D, which is equal to the " Angle D F E, is * equal to A F C, the Angle A F C " being vertically oppoſite to D F E. Wherefore, ſeeing " the Angle A F C is the Angle of Incidence, which in " ſuch Caſes is equal to the Angle of Reflexion, it is ma- " nifeſt that B F D, which hath been proved equal to A F C, " is the Angle of the Reflexion of the Ball A, and that *Fig.* 79. * *Per* 15. *l.* 1.

" the

" the Ball tending towards E, is in the Point F ſo reflected, " as to hit the Ball B. *Q. E. D.*

Scholium, or Obſervation.

BY much the ſame way of Reaſoning, whereby this 4th *Propoſition* has been demonſtrated, the following *Theorem*, which we ſhall have occaſicn to uſe by and by, may be demonſtrated alſo.

Fig. 25. If in two Triangles X, Z, the Sides B C and F I ſhall be equal, and the Angles adjacent to theſe two Sides equal alſo, *viz.* B and C equal to F and I; all the other Things, and the whole Triangles themſelves will be equal.

For the Side F I laid upon the Side B C will agree, or throughly coincide with it (*a*). And then becauſe the Angles B and C are equal to thoſe F and I, when the Side F I is laid upon the Side B C, F L (*b*) will fall exactly upon B A, and I L upon C A. Therefore the Point L will fall upon the Point A (for if it fall without A, the Sides F L, I L would not fall upon the Sides B A, CA.) Therefore all Things are equal by the 7th *Axiom.*

(a) *Per Axiom* 8.
(b) *Per Axiom* 8.

PROP. V. Theorem.

Fig. 26. *IN an Iſoſceles or Equicrural Triangle, the Angles at the Baſe (A, C) are equal.*

Let the Triangle A B C be underſtood to be twice put, but in an inverted poſture *c b a.* Becauſe therefore, in the two Triangles A B C, *c b a* the Side A B is by the Suppoſition equal to the Side *c b*, and the Side C B to the Side *a b*, and the Angle B to the Angle *b*; the Angle A alſo at the Baſe will (*c*) be equal to the Angle *c*. *Q. E. D.* For as for the Angles C and *c* they are the ſame.

(c) *Per* 4. *l.* 1.

Corollary.

THEREFORE an Equilateral Triangle is alſo Equiangular.

PROP.

PROP. VI. Theorem.

IF in a Triangle (A B C) two Angles (A and C) be equal, the Sides also (A B, B C) which are opposite to those Angles are equal also. Fig. 6.

Let the Triangle A B C be supposed to be twice put but in an inverse Situation, *c b a*. Because therefore in the Triangles A B C, *c b a*, one Side A C is equal to one Side (*c a*) and the Angle A is equal to the Angle *c*, and the Angle C equal to the Angle *a*, all the other Things shall be likewise (*a*) equal, and consequently A B shall be equal to the Side *c b*. Q. E. D. For as for the Lines C B and *c b* they are the same.

(a) *Per Schol. Prop.* 4.

Corollary.

THEREFORE an Equiangled Triangle, is also Equilateral.

Corollary (2.) " Hence, by the means of the shadow of the Sun, we may measure the height of a Tower, or any elevated Point. For when the Sun is elevated 45 Degrees above the Horizon, the Shadow which the Tower casts towards the Horizon will be exactly equal to its Height. For, by reason that the Angle A C B is half a right Angle, the Angle B A C also * will be half a right one; and so, by the Force of the present Proposition, the Line A B will be exactly equal to the Line B C. The Line B C therefore being found by measuring, there is found at the same time the Line A B, the Height of the Tower above the Horizon. *Fig.* 80.

* *Per Corol.* 11. *Prop.* 32. *l.* 1.

Corollary (3.) " The same Thing also may be found without the Sun by the means of an Astronomical Quadrant. For where the Angle of Elevation is half-right, there the Height of the tower above the Observer's Eye is equal to the distance of the same Eye, from that part of the Tower which is opposite to it. The distance therefore of the Eye from the Tower being given by measuring, there is given at the same time the Height of the Tower. *Q. E. I.*

The

The Seventh Propoſition in *Euclid* is for the ſake of the Eighth, which without it will here be demonſtrated.

PROP. VIII.

Fig. 27. *IF two Triangles (X, Z) have all their Sides equal amongſt themſelves reſpectively (AC equal to EF; CB to FI; AB to EI;) they will alſo have all the Angles which are oppoſite to equal Sides, equal: (C equal to F; A to E; B to I.)*

For ſuppoſe the Side A B laid upon its Equal E I, if then the Point C falls upon F, the Triangles will in the Whole agree or coincide, and conſequently all the Angles will be equal. But the Point C will fall upon the Point F. For,

Fig. 81. " From the Center A let a Circle be deſcribed with the " Semi-diameter E F; and from the Center I, let another " Circle be deſcribed with the Semi-diameter I F; the " Point C by reaſon of the Equality of the Sides of both " Triangles, will be in the Circumference of both Circles, " and conſequently in the Point E, the common Interſec- " tion of both theſe Circumferences. *Q. E. D.*

PROP. IX. Problem.

Fig. 29. *TO Biſect or Divide into two equal Parts, a given right lin'd Angle, as IAL.*

From the Sides of the Angle take with a pair of Compaſſes two equal Lines, A B, A C; then from the Centers B and C deſcribe two equal Circles cutting one another in F; which done, draw the Line F A. This biſects the Angle.

For draw the Line B F, C F; the Triangles F A B, F A C are to each other Equilateral; for the Sides A B, A C are by the Conſtruction equal, as in like manner are the Sides B F, C F, they being Semi-diameters of equal Circles; and A F is common to both Triangles. Therefore the Angles B A F, C A F (*d*) are equal. Therefore the given Angle I A L is biſected. *Q. E. F.*

(d) *Per* 8. *l.* 1.

Corollary.

Corollary.

HENCE we may learn how an Angle may be divided into all equal Angles, 4, 8, 16, *&c. viz.* by bisecting each part again.

Scholium.

NO one hath hitherto taught the Way of dividing Angles into all equal Parts whatsoever with a pair of Compasses and a Rule.

Yet may you divide any given Angle mechanically into any equal Parts whatsoever, if from the top of the Angle, as the Center, you describe an Arch between the Legs of the Angle, and divide the Arch into as many equal parts as you require; for right Lines let down from A, through the points of the Division, will cut the Angle into so many equal parts. *Fig.* 30.

PROP. X. Problem.

TO *Bisect a finite given Line* (*AB.*) *Fig.* 31.

Upon the given Line A B make an Equilateral (*a*) Triangle A G B, bisect its Angle G (*b*) with the right Line G C. The same shall bisect the given Line A B.

(a) *Per* 1. *l.* 1.
(b) *Per preced.*

For in the Triangles X, Z, the Side C G is common; and by the Construction G B, G A are equal, and the Angles contained between them A G C, B G C, are likewise equal. Therefore the Bases A C, B C (*c*) are equal. The given Line therefore A B is bisected. *Q. E. F.*

(c) *Per* 4 *l.* 1.

But for practice it is sufficient, from the Centers A and B, to describe two equal Circles, cutting one another in G and L, and so to draw the right Line G L.

PROP. XI. Problem.

FROM *a given Point* (*A*) *to raise a Perpendicular in a given right Line L I.*) *Fig.* 32.

With a pair of Compasses take the equal Lines A C, A F. From the Centre C and F describe two Circles, with the same radius

cutting

cutting one another in B. The Line which is drawn from B to A will be the Perpendicular required.

For let the right Lines C B, F B be drawn. The Triangles X and Z are Equilateral to one another. Therefore

(a) *Per* 8. *l.* 1. the Angles C A B, F A B are equal (*a*) Therefore B A is (*b*) perpendicular to the Line (L I.) *Q. E. F.*

(b) *Per Def.* 14. In Practice this and the next are easily preformed by the help of a Square.

PROP. XII. Problem

Fig. 33. *FROM a given Point (A) which is without an infinite right Line (as L Q) to let fall a Perpendicular to that Line.*

From the Center A describe a Circle which may cut the

(c) *Per* 10. *l.* 1. given L Q in C and I. Bisect the right Line C I (*c*) with the right Line A B. This A B is the Perpendicular required.

For let there be drawn A C. A I. Because by the Construction, the Triangles X and Z are Equilateral to one a-

(d) *Per* 8. *l.* 1. nother, Therefore the Angles (*d*) C B A, I B A, are equal. Therefore A B is (*e*) Perpendicular. *Q. E. F.*

(e) *Per Def.* 14.

PROP. XIII. Theorem.

Fig. 34. *THE right Line (B A) standing upon the right Line (C F) either makes two right Angles, or Angles equal to two right ones.*

For if B A stand upon it perpendicularly, then by the 14th *Definition*, the two Angles B A C, B A F will be right ones. And if B A stand obliquely, let there be rais'd

(f) *Per* 11. *l.* 1. (*f*) the Perpendicular A L. Where, because the unequal Angles C A B, F A B possess the same place which the two right ones C A L, L A F do, and agree to them, they

(g) *Per Axiom* 7. are equal (*g*) to them. *Q. E. D.*

Corollaries.

1. IN the same manner it will be demonstrated if more right Lines than one stand upon the same right Line, that the Angles thereby made are equal to two right ones.

2. Two

2. Two right Lines cutting one another, B A C, FA L, make the Angles equal to four right ones. *Fig.* 37.

3. All the Angles which are about one Point, make Angles equal to four right ones. It appears from *Corollary* 2. for they are four right ones, cut into more Parts. *Fig.* 36.

4. The Angle CAF being known, you at the same time know its Compliment unto two right Angles B A F. For Example, Let the Angle C A F be of 70 Degrees; the Angle B A F will be of 110 Degrees. For those two Numbers added together make 180 Degrees, which is the Measure of two right Angles. *Fig.* 37.

PROP. XIV. Theorem.

If two right Lines (X R, Z R) at the same Point of a right Line (Q R) make the Angles on both Sides (X R Q, Z R Q) equal to two right Angles; the Lines (X R, Z R) make one right Line. *Fig.* 35.

If you deny it, Let X R, B R make one right Line. Therefore the Angles X R Q, Q R B (*a*) will make two right Angles. Which thing is (*b*) absurd; seeing by the Hypothesis X R Q, Z R Q do make two right Angles.

(a) *Per* 13. *l.* 1

(b) *Contra. Axiom* 9.

PROP. XV. Theorem

If two right Lines (B C, F L) cut one another in A, the Angles opposite at the top (A) are equal, viz. L A B to C A F, and B A F to L A C. *Fig.* 37.

For because B A stands upon the right Line L F, the Angles L A B, F A B are (*c*) equal to two right ones: And because F A stands upon the right Line B C, the Angles F A C, F A B are also equal (*d*) to two right ones. Therefore the two Angles together (*e*) L A B, F A B are equal to those two together C A F, F A B; by taking away F A B, common to both, L A B (*f*) remains equal to C A F. In the same manner B A F, L A C are shewed to be equal.

(c) *Per* 13. *l.* 1.

(d) *By the same Prop.*

(e) *Per Axiom* 1.

(f) *Per Axiom* 3.

Corollary. " From these two Propositions we gather in " Catoptrics, that a Ray of Light, as reflected in an Angle

" equal to the Angle of Incidence, taketh the ſhorteſt way
Fig. 82. " of all. *e. g.* When the Angles B E D, A E F are " equal, the Lines A E and E B taken together, are " ſhorter than any Lines whatſoever, as A F and F B taken " together. For from the Point B, let the perpendicular " Line B C be let down; and let B D and D C be equal: " Let the Lines alſo E C and F C be drawn. Now in the " Triangle B E D and D E C, ſeeing the Side D E is com- " mon to both, and the Side B D and D C are equal by " the Hypotheſis, as is alſo in the like manner BDE equal
* *Per* 4. *l.* 1. " to the Angle C D E; the Triangles alſo ſhall be * equal " in all other Things, and B E ſhall be equal to C E, and " the Angle B E D to the Angle D E C; (where, becauſe " the Angle D E C is equal to [B E D, that is] A E F, the " Lines A E, E C are proved to make one right Line.) " And in the ſame manner the Line B F will be proved " equal to F C. Seeing therefore the Lines B E and E A " taken together, are equal to the Line C A, and the Lines " B F, F A taken together, are equal to the Lines C F, F A " taken together; it is manifeſt that C A, which is one
* *Per* 20. *l.* 1. " Side of the Triangle A C F * is leſs than the two Sides " C F, F A taken together. *Q. E. D.*

PROP. XVI, XVII.

THESE two Propoſitions are contained in Propoſition 32; *and are not here made uſe of till then.*

PROP. XVIII. Theorem.

Fig. 38. *IN every Triangle the Angle* (*A*) *which is oppoſed to the greater Side* (*BO*) *is the greater; and* (*B*) *which is oppoſite to the leſſer Side* (*AO*) *is the leſſer Angle.*

(A) cannot be equal to (B) for then the oppoſite Sides
(*a*) *Per* 6. *l.* 1. B O, A O would be equal (*a*); which is contrary to the Hypotheſis. Neither can A be leſs than B, for if it were ſo, there might within the Angle B be made an Angle A B F by the right Line B F; which Angle ſhould be equal to A. But then by the 6th of this Book B F, A F ſhall be equal;

and

and if you add to both O F, then B F, F O ſhall be equal to A O. But A O by the Hypotheſis is leſs than B O. Therefore B F, F O ſhall be leſs than B O, which contradicts the Definition of a right Line, which is the ſhorteſt of all betwixt two Points. Therefore the Angle A is neither leſs than B, nor equal to it. Therefore it is greater. *Q. E. D.*

PROP. XIX. Theorem.

IN the Triangle A O B the Side (B O) which is oppoſed to the greater Angle (A) is the greater; And (A O) which is oppoſed to the leſſer Angle B, is the leſſer. *Fig.* 38.

This Propoſition is the converſe of the former. B O is not leſs than A O, for if it were, the Angle (A) by the 18th would be leſs than B; which is contrary to the Hypotheſis. Nor can B O be equal to A O, for in this caſe, by the 5th, the Angles A and B would be equal. But this Equality of thoſe Angles is contrary to the Hypotheſis. Therefore B O is greater than A O. *Q. E. D.*

Corol. " Hence we gather, that a Globe, or Ball per- *Fig.* 83.
" fectly poliſhed, cannot reſt in an horizontal Plane per-
" fectly poliſhed, but where it toucheth the Earth. For
" let the Line A B be an horizontal Plane, C the Earth's
" Center, C A the Semi-diameter of the Earth, perpen-
" dicular to the Tangent A B. The Globe placed at B,
" becauſe of its Gravity, and the Declivity of the Plane,
" will deſcend towards A. For in the Triangle C A B, the
" perpendicular Line C A, which is oppoſite to the acute
" Angle A B C, is leſs than the Line B C, which is op-
" poſed to the right Angle B A C; and ſo there is from B
" to A a perpetual Deſcent, in which the Globe cannot
" reſt. And in the like manner we prove the Deſcent of
" Fluids and their Conformation into a ſpherical Surface.

PROP. XX. Theorem.

IN any Triangle, any two Sides of it taken together, are greater than the remaining Side.

This, with *Archimedes*, is, as it were an *Axiom*; forasmuch as it is immediately manifest out of his Definition of a right Line; which see above amongst the Definitions.

PROP. XXI. Theorem.

Fig. 39. *IF from the Ends of one Side A B, two right Lines be drawn, and joined together within the Triangle, (as the Lines A O, B O) these are less than the Sides of the Triangle (A C, B C) but they comprehend a greater Angle (A O B.)*

For as for the first Part of the Proposition, draw out A O unto F: A C, C F are (*a*) greater than A F. Therefore the common Line F B being added, A C, B C are greater than A F, F B. Again, OF, F B are greater (*b*) than O B. Therefore the common A O being added, A F, B F are greater than A O, B O. Therefore A C, C B are much greater than A O, O B.

(a) *Per* 20. *l.* 1.

(b) *By the same.*

The second Part of this Proposition will be demonstrated in the second Corollary of the first Part of Proposition 32. And in the mean while we shall make no use of it.

PROP. XXII. Problem.

Fig 40. *TO make a Triangle of three given right Lines (B O, L B, L O) of which any two must be greater than the third.*

Let B L, one of the given Lines be taken, and B one of its Extremities being taken for the Center, with the Interval of the other given Line B O, describe an Arch.

Then the other Extremity L being taken for the Center, with the Interval of the third given Line L O, describe an Arch, cutting the former in O; which being done, and the right Lines B O, L O being drawn, I say that that is done, which was to be done.

The Demonstration is manifest from the Construction.

PROP.

PROP. XXIII. Problem.

AT a given Point in a right Line (as B) to make an Angle equal to a given one (A.)

First of all let C F be drawn at a venture, cutting the Sides of the given Angle A. Then in the given right Line from B, take B L equal to A F. Then from the Center B describe a Circle with the Interval A C; afterwards another from the Center L, with the Interval F C, which may cut the former in O. Then from O unto B, and L having drawn right Lines, the Angle L B O will be equal to the given one A. *Fig.* 40.

For by the Construction, the Triangles are Equilateral to one another. Therefore by the 8th of this Book the Angles B and A are equal.

Scholium.

IT seems meet for the sake of Beginners to propound some things here which are necessary for Practice about Angles.

The Measure of an Angle is the Arch of a Circle, which is described from A, the Top of the Angle as the Center. Therefore look how many Degrees the Arch B C, which is intercepted between the Legs of the Angle B A C shall contain, of so many Degrees the Angle B A C shall be said to be And so because B F, a quarter of the Circumference, contains 90 Degrees, and measures the right Angle B A F, a right Angle shall be said to be of 90 Degrees. In like manner, because half the Circumference, which is divided into 180 Degrees, measures two right Angles, and the whole Circumference, which is divided into 360 Degrees, measures four right Angles; two right Angles shall be said to make 180 Degrees and four 360 Degrees. These Things being premised, the Practice about Angles is as follows. *Fig.* 41. *Fig.* 42.

1. At B a given Point in a right Line to make an Angle equal to the given one A.

From A, the Top of the given Angle as the Center, describe betwixt the Sides the Arch C F. Then from B, the given Point as the Center, describe with the same Interval the Arch L Z; from which take off L O equal to CF. Through B and O draw a right Line; L B O shall be equal to the given A.

Fig. 3. 2. To examine the Degrees of the given Angle O P Q. This is done very easily through any Semi-circle or Protractor, which is divided into 180 Degrees. For put the Center of the Semi-circle upon P, the top of the Angle, and the Radius of the Semi-circle P L upon the Side of the Angle P Q; and the Arch L O, which is intercepted betwixt the Legs of the Angle, will shew of how many Degrees the given Angle is.

3 To frame an Angle, containing a given Number of Degrees, as 42.

Fig. 43. Draw the right Line X Q, in which mark the Point P. Upon P put the Center of a Semi-circle, and its Semi-diameter P L upon P Q. From L number 42 Degrees, that is, until you come to O. A right Line drawn from P through O, will give the Angle O P L of 42 Degrees.

PROP. XXIV, XXV. Theorems.

Fig. 44. *If two Triangles (B A C, B A F,) shall have two Sides (B A, A C,) equal to two (B A, A F,) one Side of one, to one Side of the other; and if one of the Triangles hath the Angle B A F) contained by those Sides greater than the other (B A C,) it shall have the Base B F greater than the Base (B C.)*

And again, if it hath the Base greater, it shall have the Angle greater.

From the Center A, describe a Circle which passeth through C, it shall pass also through F, because A C, A F are supposed to be equal. Therefore B F shall fall betwixt the Point A and C.) Then join C F. The Angle B C F is greater than the Angle A C F; that is, by the 5th of this Book, than the Angle A F C, and consequently much greater than the Angle B F C. Therefore in the Triangle B C F, (a) B F, which is opposite to the greater Angle B C F, is greater than B C, which is opposite to the lesser Angle B F C.

(a) Per 19. l. 1.

2. As for the second Part of the Proposition, this is manifest from the first Part, and Proposition 4.

PROP.

PROP. XXVI. Theorem.

IF two Triangles (X and Z) have two Angles equal to two, one Angle of the one, equal to one Angle of the other (B to F and C to I,) and one Side of one Equal to one of the other, whether it be that which is betwixt the equal Angles (as BC=FI) or a Side which is opposed to one of the equal Angles (as AC=LI,) all the other Parts shall be equal. Fig. 25.

For in the Place, let the Sides (BC, FI) which are betwixt the equal Angles, be supposed equal: In this Case all the other Parts are equal: as hath been already demonstrated in the *Scholium* of the 4th Proposition.

Again, suppose the Sides AC, LI, which are opposed to the equal Angles, to be equal. Here, because the Angles (B, C) are by the Hypothesis equal to (F, I) the other Angles, also (A, L) shall be equal by *Corollary* 9. *Proposition* 32. which Proposition depends not upon this Therefore by the first Part of this, all the other Parts are equal.

Corollary. " Hence also, following *Thales*, we may Fig. 84.
" measure inaccessible Distances. *e. g.* Let AD be an
" inaccessible Line; to which at the Point A, let there be
" erected the Perpendicular AC. Let there be made the
" Angle (ACB) equal to the Angle (ACD) the accessible
" Line AB shall be equal to the inaccessible AD. *Q. E. I.*

PROP. XXVII. Theorem.

IF the right Line GO shall cut two right Lines which are parallel (AB, CF,) 1. The alternate Angles (RLO, QOL, likewise BLO, COL) shall be equal. 2. The external Angle GLB shall be equal to the internal one on the same Side (that is, to LOF) as likewise GLR equal to LOC. 3. The two internal ones on the same Fig. 45.

ſame Side (ALO, COL) as taken together, ſhall be equal to two right ones, as likewiſe the two (BLO, FOL) equal to two right ones.

Fig. 46. The firſt Part is thus proved. From O and L draw the Perpendiculars O R, L Q. Theſe are perpendicular to the

* *Per Axiom* II. * two Parallels A B, C F; and by Definition 36, equal betwixt themſelves, they ſhall therefore (*a*) intercept equal

(a) *Per Axiom* 12. Parts of the Parallels, and R L ſhall be equal to Q O. Therefore the Triangles X and Z are Equilateral to one

(b) *Per* 8. *l.* 1. another. Therefore (*b*) the alternate Angles R L O, Q O L, which are oppoſite to the equal Sides R O, Q L, are equal. Which is the firſt Thing. From whence it is likewiſe manifeſt, that the Alternates B L O, C O L are equal. For becauſe, as well B L O, A L O, as C O L, F O L are equal

(c) *Per* 13. *l.* 1. (*c*) to two right ones; therefore B L O, A L O together, are equal to C O L, F O L. Therefore taking away the Equals R L O, F O L, the remaining ones B L O, C O L, ſhall be likewiſe equal.

Part ſecond. The Angle G L B is equal to that which

(d) *Per* 15. *l.* 1. is vertically oppoſite R L O (*d*). But R L O, by the firſt Part of this Propoſition, is equal to L O F. Therefore G L B, the external Angle, is equal to the internal remote one, which is on the ſame Side, L O F.

Part third. A L O, by the firſt Part, is equal to L O F.

(e) *Per* 13. *l.* 1. But LOF, with COL, make (*e*) Angles equal to two right ones. Therefore A L O, with C O L, doth the ſame.

Fig. 31. Corol. "Hence, in Imitation of *Eratoſthenes*, we learn "to meaſure the Compaſs of the Earth. For he obſerved, "that on the Day of the Summer Solſtice, the Sun was "perpendicularly over *Siene*, a City of *Egypt*; and he "found by the means of a Stile, perpendicularly erected, "that on the ſame Day the Sun was diſtant from the ver- "tical Point of *Alexandria*, a City of *Egypt*, ſituate almoſt "under the ſame Meridian with the other, ſeven Degrees, "with one Fifth Part of a Degree; and he knew that "theſe two Cities were about 5000 Furlongs diſtant from "each other. From theſe Things, by the help of this "Propoſition, he determin'd the Compaſs of the Earth. "Let A be *Siene*, and B be *Alexandria*, where the Gnomon "B C is erected perpendicular to the Horizon. Let D F

and

" and E G be the Solar Ray's parallel to one another as to " Senfe. D A a Ray perpendicular to the Horizon of *Si-* " *ene*; and E G a Ray oblique to the Horizon of *Alex-* " *andria*, and which paffing by the top of the Gnomon, " makes with it the Angle G C F, which is of $7\frac{1}{5}$ Degrees. " Now feeing the Angle G C F is equal to the alternate " one A F B, and the meafure of it is the Arch A B of " $7\frac{1}{5}$ Degrees; he found the Compafs of the Earth by " this Analogy, as $7\frac{1}{5}$ Degrees are to 5000 Furlongs; fo " the whole Circumference, which is of 360 Degrees, is " in a grofs Number to 250000, the Compafs of the Earth " in the fame meafure. *Q. E. I.*

PROP. XXVIII. Theorem.

If a right Line (G O) cutting two right Lines (A B, C F) makes the alternate Angles (A L O, L O F) equal; the Lines (A B, F C) are parallel. Fig. 47.

If you deny it, let X L Z, paffing through the Point L, be parallel to C F. Therefore X L O (*a*) is equal to the alternate F O L, which cannot be, feeing by the Hypothefis A L O is equal to F O L.

(a) *By the foregoing.*

PROP. XXIX. Theorem.

If a right Line (G O) cutting two right Lines (A B, C F) fhall make the external Angle (G L B equal to the internal oppofite one (L O F,) or fhall make the two internal Angles on the fame Side (A L O, C O L) equal to two right Angles; (A B, C F) are parallel Lines. Fig. 45, 46.

By the 15th of this Book, G L B is equal to A L O, which is vertically oppofite to it. But by the Hypothefis G L B is equal to L O F. Therefore alfo A L O is equal to its alternate one L O F. Therefore (*b*) A B, C F are parallel.

(b) *By the foregoing.*

Again, C O L with F O L makes Angles equal to two right ones. But by the Hypothefis C O L with A L O, makes in all two right Angles alfo. Therefore A L O,

F O L,

F O L, the alternate Angles are equal. Therefore again, (a) A B, C F are parallel.

(a) *By the foregoing.*

Corollary. " From the fecond Part of this Propofition " it appears that every Rectangle is a Parallelogram.

PROP. XXX Theorem.

Fig. 45.

IF two right Lines (A B, C F) be parallel to the fame right Line (D N) they are parallel betwixt themfelves.

It is manifeft in itfelf, and from the foregoing Propofitions. For if all be cut by the right Line G O, the external Angle G L B is equal (*b*) to the internal oppofite one L D N. Now L D N is an external Angle in refpect of D O F, and therefore (*c*) equal to it. Therefore alfo G L B is equal to L O F. Therefore A B, C F (*d*) are parallel.

(b) *Per* 27. *l.* 1.
(c) *By the fame.*
(d) *By the foregoing.*

PROP. XXXI. Problem.

Fig. 48.

THROUGH a given Point (A) to draw a Parallel to a given right Line (F C.)

From the Point A, let there be drawn at random A L, cutting the given F C. At the Point A, let there be made the Angle (*e*) L A S equal to the Angle A L F. The Line A S will be parallel to C F; as is manifeft from the 28th, the alternate Angles S A L, A L F being equal.

(e) *Per* 23. *l.* 1.

As for the practice. Draw A L, and from the Center L defcribe an Arch I Q; and from the Center A, with the fame Interval, defcribe the Arch O X; from which, having taken off O B equal to I Q, the right Line drawn through A and B will be the Parallel fought. The Demonftration depends upon the 29th, *l.* 1.

Fig. 49.

Or otherwife thus. From a certain Center P defcribe a Circle which may pafs through the given Point A, and may cut the given Line C F in Q and O. Take the Arch O N equal to Q A. The right Line A N fhall be the Parallel fought.

The Demonftration hereof depends upon the 29th, *l.* 3. and the 28th of this.

PROP.

PROP. XXXII. Theorem.

PART I.

IN every Triangle any one of the external Angles (as F B C) is equal to the two internal remote ones (A and C.) Fig. 51.

Through the Point B draw (*a*) B L parallel to A C. Becauſe F A cuts the two Parallels B L, A C, the external Angle F B L ſhall be equal to the internal one A (*b*.) And becauſe the Line B C cuts the ſame Parallels (B L, A C;) the Angle L B C ſhall be (*c*) equal to its alternate one C. Therefore the whole Angle F B C ſhall be equal to A and C both together. *Q. E. D.*

(a) *Per* 31: *l.* 1.
(b) *Per* 27. *l.* 1.
(c) *By the ſame.*

Corollaries.

1. THE external Angle F B C is greater than either of the internal oppoſite ones A or C. Fig. 51.

2. Of the Angles (C and A O B) having the ſame Baſe, A B which falls within, is the greater. Fig. 39.

For let A O be produced unto F, A O B, by this Propoſition, is greater than O F B; and likewiſe O F B is by this greater than C. Therefore A O B is much greater than C. Fig. 55.

3. If from one Point A there falls two right Lines upon B C; one of them A O obliquely, the other A F perpendicularly; this laſt ſhall fall on the Side of the acute Angle AOB. For let it fall, if it may be, on the Side of the obtuſe Angle A O C; as for inſtance, in Q. In this Caſe, the acute Angle A O B ſhall be external in reſpect of A Q B, and conſequently ſhall be greater than the right one, by *Corollary* 1, which is abſurd.

PROP.

PROP. XXXII. Theorem.

PART II.

IN every Triangle the three Angles taken together, are equal to two right ones, and therefore make 180 *Degrees.*

Fig. 52. Draw forth one Side A B unto F. The external Angle F B C is equal (*a*) to the two internal oppoſite ones, A and C. But F B C with A B C, make (*b*) Angles equal to two right ones. Therefore the two, A and C, with the ſame CBA, make Angles equal to two right ones. *Q. E. D.*

(a) *By the firſt Part of this.*

(b) *Per* 13. *l.* 1.

Fig. 53. Or thus. Draw the Line H M parallel to A C, the alternate Angles, as well O and A, as N and C (*c*) are equal. But O, Q, N make Angles (*d*) equal to two right ones. Therefore alſo A, C, Q are equal to two right ones. *Q. E. D.*

(c) *Per* 27. *l.* 1.

(d) *Corol.* 1. *Prop.* 13. *l.* 1.

Corollaries.

4. THE three Angles of any one Triangle taken together, are equal to the three Angles of any other Triangle taken together

5. If in a Triangle one Angle be right (or obtuſe) the reſt are acute.

6. If in a Triangle one Angle be right, the two other Angles together make one right Angle

7. In every Triangle, the Angle which is right, is equal to the other two taken together.

8. When you know of how many Degrees one Angle of a Triangle is, you know at the ſame time how many Degrees the two other Angles, as taken together, do make up. And ſo on the contrary, when you know how many Degrees two Angles of a Triangle take together do make up, or what is the Sum of them, you know at the ſame Time of how many Degrees the third Angle is.

9. When two Angles of one Triangle, either ſeverally or together, are equal to two Angles of another Triangle; the third Angle of one Triangle is alſo equal to the third of the other.

10. When two Triangles have one equal Angle, the Sum alſo of the reſt of the Angles are equal.

11. When

11. When in an *Isosceles*, the Angle contained by the equal Sides is a right one, the two other are, each of them, half-right Angles. And the Angles of an *Isosceles*, which are at the Base, are always acute.

12. In an Equilateral Triangle each Angle is two thirds of a right Angle. For it is one third of two right ones, therefore it is two thirds of one right one.

13. Hence a right Angle (B A C) is easily divided into three equal parts; if upon A C be made the equilateral Triangle Z; for seeing F A C is two thirds of one right one, BAF shall be one third of a right one. *Fig.* 54.

14. The Perpendicular A F is the shortest of all Lines which can be drawn from the Point (A) unto some right Line. For seeing the Angle F is a right one, AOF shall, by the 5th *Corollary*, be an acute one. Therefore (a) A F is shorter than any other, as A O. *Fig.* 55.

(a) *Per* 19. *l.* 1.

25. Only one Perpendicular can fall from one Point unto one right Line. This is manifest out of the foregoing *Corollary*.

16. " Hence also we learn to determine the Parallax of the Stars, or the difference of their true and apparent Place. Let A be the Center of the Earth, B the Place of the Observer upon the Surface of the same. Let D B C be the Angle of the Star C, according to Observation, or the *visible* Angular distance thereof from the vertical Point; when in the mean while D A C is the *true* Angular Distance. Now the external Angle D B C, which is given from Observation is equal to the Angles B A C and B C A, taken together; and consequently the Angle B C A is the difference of the Angles D B C and D A C. If therefore we shall from Astronomical Tables seek the Angle D A C, or what at that Time of Observation is the true Angular Distance of the Star from the vertical Point, when the Angle D B C is at the same Time known by means of the Quadrant, the Difference of those Angles B C A, which we call the *Parallax*, will likewise be known. *Q. E. I.* *Fig.* 86.

Scholium.

BY the Testimony of *Eudemus*, an ancient Geometrician, *Pythagoras* was the Finder-out of this Proposition, which indeed is a Theorem most excellent in it self,

ſelf, moſt fruitful in its Confectaries, and of immenſe uſe in all Parts of the Mathematicks. *Ariſtotle* very frequently makes mention of it, who alſo puts it for an Example of the moſt perfect Demonſtration. But like, as from this Propoſition, we have already learned, how many right Angles, the Angles of a Triangle are equivalent to; ſo by the help of the ſame, it will in the three following Propoſitions be manifeſt, how many right Angles, the Angles of any Rectilinear Figure whatſoever, whether internal or external, do make.

Theorem 1.

Fig. 56. IN every Quadrangular Figure, the four Angles together make four right ones.

For if, through the oppoſite Angles, you draw the right Line B F, this will cut the Quadrangle into two Triangles, without forming any new Angles, whoſe Angles together do (*a*) make four right Angles.

(a) *Per* 32. *l.* 1.

Theorem 2.

ALL the Angles together of every right-lin'd Figure make twice ſo many right ones, abating four, as are the ſides of the Figure.

Fig. 57. From any Point A within the Figure, let there be drawn unto the Angles of the Figure right Lines, which ſhall cut the Figure into ſo many Triangles as it hath Sides, and make no more Angles, but thoſe of the Center. Wherefore, when each of the Triangles contains two right Angles (*b*), they muſt all together contain twice ſo many right Angles as there are Sides. Now the Angles about the Point A (*c*), do make four right Angles. Therefore, if from the Angles of all the Triangles, you take away the new Angles which are about A, the remaining Angles, which indeed do alone conſtitute the Angles of the Figure, will make twice ſo many right Angles, excepting four, as are the Sides of the Figure.

(b) *Per* 32. *l.* 1.
(c) *Corol.* 3. *Prop.* 13. *l.* 1.

Hence it appears, that all Right-lin'd Figures of the ſame Species, or Number of Sides and Angles, have the Sum of their Angles equal. Which thing is worthy of admiration.

The Practice is thus; Double the Denominator of the Figure, and from the Product take away four; the Remainder

mainder is the Number of the right Angles, which the internal Angles of the Figure do make.

Theorem 3.

ALL the external Angles of any Right-lin'd Figure whatsoever taken together do make up four right Angles. *Fig.* 58.

For each of the internal Angles of the Figure does (*d*) with its respective external one, make two right Angles. Therefore all the internal ones, together with all the external ones, do make up twice so many right Angles as are the Sides of the Figure. Now, by the Preceding, the internal ones, together with four right Angles added to them, make twice so many right Angles as are the Sides of the Figure. Therefore the external Angles are equal to four right ones.

(d) *Per* 13. *l.* 1.

Wonderful truly is this Property of Right lin'd Figures; from whence it follows also, that all the Right-lin'd Figures of any Species whatsoever have the Sums of their external Angles equal. And therefore the three external Angles of a Triangle are equal to the thousand external Angles of a thousand-sided Figure. Which Observation is altogether worthy of Admiration.

PROP. XXXIII. Theorem.

IF two right Lines, which are equal and parallel, as (AB, CF) be joined by two others, (AC, BF;) these will also be equal and parallel. *Fig.* 59.

Let A F cut the Parallels A B, C F. In the Triangles Q, R, the alternate Angles B A F, C F A (*a*) will be equal. Now the Side A B is supposed equal to the Side C F, and A F is common to both Triangles. Therefore (*b*) the Bases B F, A C are equal. (Which is the first Part.) And then the Angles at the Bases A F B, F A C are equal; which being made by A F falling upon the right Lines A C and B F, are alternate Angles A F B, F A C equal. Therefore A C, B F are also (*c*) parallel. Which is the other Part.

(a) *Per* 27. *l.* 1.

(b) *Per* 4. *l.* 1.

(c) *Per* 28. *l.* 1.

Corollary.

Corollary. 1. " Hence we learn to measure as well " the Heights of Mountains above the Horizon as their " horizontal Lines. Let A B C be the Side of a Mountain, " to which apply a great Square, or some Instrument " equivalent thereto A D B. Then shall A D be equal to " H B, and D B equal to A H. Then coming unto the " lower Part, which is from the Point B unto the Point C, " practise as before. So shall E B be equal to C F, and " E C be equal to B F. Which done, the Sides parallel " to the Horizon, A D, B E, *&c.* added together will " give the horizontal Line G C; and the perpendicular Sides " B D, E C, *&c.* added together, will give the Height A G.

Fig. 59. Corollary (2.) " Hence also we learn to estimate the " Composition of Motions. Let a Body placed at A be " driven in the same Moment of Time by the Force A C, " according to the Direction of the Line A C, and by the " Force A B, according to the Direction of the Line A B. " From the Conjunction of these two Forces it will de- " scribe the Diagonal A F. For in this Line of its Mo- " tion, neither of the Forces is changed: For the Body at " F is equally distant from both the Lines of Direction " A C, A B, as if it had been driven by either of the " Forces separately; which thing can be said of no other " Point. And this Corollary doth so fully agree with " Astronomical and other Mechanical Phænomena, that it " is worthily reckoned by the famous Sir *Isaac Newton* " as a Foundation of his Geometrical Philosophy.

PROP. XXXIV. Theorem.

Fig. 59. *IN every Parallelogram the opposite Sides an[d] Angles are equal, and it is cut into tw[o] equal Parts by the Diameter.*

(a) *Per Def.* 35. (b) *Per* 27. *l.* 1. (c) *Per Def.* 35. (d) *Per* 27. *l.* 1.

Because A B, C F are (*a*) parallel, and A F falls upo[n] them, the alternate Angles B A F, C F A are (*b*) equa[l]. Likewise because A C, B F (*c*) are parallel, and upo[n] them falls the Line A F, the Alternates C A F, B F A ([*d*]) are equal. Therefore the whole Angle B A C is equal [to] the whole Angle B F C. In the same manner B and C a[re] shewed to be equal. Which was the first Part.

Now because it hath been already shewed, that t[he] Triangles Q, R, which have one common Side A [F] h[ave]

have alſo the Angles adjacent to the common Side equal, B A F to C F A; and C A F to B F A; the Sides likewiſe ſhall be equal, (a) A B to F C, and B F to A C; and thus the whole Triangles are equal. Which was the ſecond Part. (a) *Per* 26. *l.* 1.

Scholium.

PARTLY from this Theorem, and partly from a Definition to be premiſed to the ſecond Book, the meaſuring of a right-angled Parallelogram is eaſily deduced. The Area thereof being produced by the Multiplication of the two contiguous Sides A F, A C one by another. *e. g.* Let A F be a Line of 8, A C a Line of 4 Feet. Multiply 8 by 4, there ariſes 32 Square Feet for the Area of the Rectangle. *Fig.* 60.

But the Area of a Square is had from the Multiplication of the Side F I by itſelf; as if F I be of 5 Feet, multiply 5 into itſelf, there will ariſe 25 Square Feet for the Area of the Square. *Fig.* 61.

The Demonſtration is manifeſt from this Propoſition, if parallel Lines be drawn through the Diviſions of the Sides.

Corollary. "Hence Surveyors do eaſily divide the Area of a Field, when it is a Parallelogram. For let A B C D be the Parallelogram Field; A D the Diameter, or Diagonal Line of the ſame, the middle Point whereof is marked F. Whatſoever right Line, as E G, paſſeth through the Point F, it divides the Field into equal Parts E A C G, E B D G. For the Triangle A B D is equal to the Triangle A C D, and * the Triangle A E F equal to the Triangle G F D. If therefore to the Trapezium E B D F, inſtead of the Triangle A E F, you ſhall add the Triangle which is equal to it, G F D, you will not change the Area; but the Trapezium E B D G will be equal to the Triangle A B D, or to half the Parallelogram, and conſequently to the Trapezium A E G C. *Q. E. D.* *Fig.* 88. * *Per* 26. *l.* 1.

D PROP.

PROP. XXXV, XXXVI. Theorems.

Fig. 62. *Parallelograms upon the ſame or equal Baſes (A B) and between the ſame Parallels (C Q, A X) are equal.*

(a) *Per Def.* 35.
(b) *Per* 27. *l.* 1.
(c) *Per* 34. *l.* 1.
(d) *Per* 34. *l.* 1.
(e) *Per* 4. *l.* 1.

Becauſe A L, B Q (*a*) are parallel, and C Q cuts them, the external Angle C L A ſhall (*b*) be equal to the internal one F Q B. Then becauſe, as well C F as L Q are equal (*c*) to the ſame A B, C F is equal to L Q. Add then F L to both, the whole Lines C L, F Q are equal. Moreover A L, B Q are equal (*d*.) Therefore the Triangles C L A, F Q B (*e*) are equal. Therefore taking away the common Triangle F O L, the Planes F O A C, Q B O L remain equal: To each of which Trapeziums add the Triangle A O B, the whole Parallelograms A C F B, A L Q B become equal. *Q. E. D.*

This Propoſition will be made univerſal. *Prop.* 1. *l.* 6. Beginners may here obſerve, that although of two Parallelograms which are between the ſame Parallels infinitely produced, and upon the ſame Baſe, one of them be extended unto an infinite Length, it ſtill remains but equal to the other, by the Force of the preſent Demonſtration.

[" From hence it follows, that two Cities in Magnitude " equal, may ſo much differ in Compaſs, that the Cir- " cumference of one may exceed that of the other an " hundred or a thouſand times. If, for inſtance, one be " of a ſquare Figure or Rectangular; but the other a Pa- " rallelogram, betwixt the ſame Parallels indeed with the " former, but very oblong.

" Moreover, it hence follows, that Figures of equal " Compaſs round may contain Areas vaſtly different.]

Scholium.

Fig. 62. FROM this Theorem we may learn to meaſure any Parallelogram. For the Area of it is produced from the perpendicular Altitude Q X, or C A multiplied into the Baſe A B.

For

For the Area of the Rectangle C B which is equal to that of the Parallelogram B L is made (*a*) by A C, multiplying A B. Therefore, &c. (a) *By the foregoing Scholium.*

PROP. XXXVII, XXXVIII. Theorems.

TRiangles (A C B, A L B) upon the ſame, or equal Baſes (A B,) and between the ſame Parallels (C I, A Z) are equal. Fig. 63.

Draw the Lines B F, B I parallel to the Sides A C, A L. The Parallelograms A C F B, A L I B (*b*) are equal. But the given Triangles are halves of thoſe Parallelograms (*c*.) Therefore the given Triangles (*d*) are equal. (b) *By the foregoing.* (c) *Per* 34. *l.* 1. (d) *Per Axiom* 6.

This Propoſition will be made univerſal, *Prop.* 1. *l.* 1. Let Beginners mark the ſame Thing here concerning Triangles, which we bid them to note in the foregoing Propoſition concerning Parallelograms.

Corollary (1.) " Hence Surveyors eaſily divide the Fig. 89.
" Area of a Triangular Field. Let A B C be the Field,
" and let the Baſe B C be biſected in D. The Triangles
" A B D, A D C upon the equal Baſes B D and D C, and
" having a common top A, or being between the ſame
" Parallels, are equal. *Q. E. F.*

Corollary (2.) " Hence we alſo gather, with the fa- Fig. 90.
" mous Sir *Iſaac Newton*, that the Areas which all Bodies
" whatſoever that revolve round about an immoveable
" Center, towards which they are impell'd, do deſcribe,
" are both in immoveable Planes, and are proportional to
" the Times of Deſcription. For let the Time be divided
" into equal Parts; and in the firſt equal Part of Time,
" let the Body, by the impreſs'd Force, deſcribe the right
" Line A B. The ſame Body, in the ſecond Part of Time,
" if nothing hindred, would go forward ſtrait unto c, de-
" ſcribing the Line B c, equal to A B; ſo that the Areas
" made by Lines drawn from the Center A S B, B S c (*a*) (a) *Per* 37. *l.* 1.
" would be equal. But when the Body comes unto B, let
" the Force act with one ſingle Impulſe, but a great one,
" and make the Body to deflect from B c, and to go forwards
" in the right Line B C: *i. e.* let the centripetal Force be
" in that Place, to the Force before impuls'd, as C c or B g

(b) *Per Corol.* 2. *Prop.* 33. *l.* 1.

(c) *Per* 37 *l.* 1.

(d) *Per Axiom* 1.

" is to B c; in this Case the Body will (*b*) describe the " Diagonal BC. Let there be drawn parallel to B S, the " right Line C c meeting BC in C. In the second Part of " Time completed, the Body will be found in the Point C, " in the same Plane with the first Triangle S A B. Join " S C. The Area made by a Ray drawn from the Center, " that is, the Triangle S B C will be equal to (*c*) S B c, and " consequently to the first Triangle S A B (*d*.) By the " same Argument, the Body, in the third equal Part of " Time, would, by its present Force, reach from C unto " d, so that the Line C d should be equal to the Line B c " or A B. But if the centripetal Force, whether it be " greater or less, does again act upon it in the Point C, " in the end of the third Part of Time, it will be found " somewhere in the Line D d, parallel to S C; and there- " fore, as before, supposing the said Force to be equal or " unequal to what it was before, it will be found to have " described the Diagonal C D, and will be found in the " Point D; and a Ray being drawn from the Center, the " Triangle S D C will be equal to that S d C, and conse- " quently to the others S C B, S A B, which are equal one " to the other. In like manner, if the centripetal Force " act successively in the Points D, E, F, and be the cause " that the Body, in the several Parts of Time respect- " ively, describes the Diagonals D E, E F, *&c.* the Area's " now made, as a-fore, will be in the same Plane, and Tri- " angles will be described equal to the former Triangles. " Therefore in equal Times, equal Area's are described in " an immoveable Plane; and so the Sums of the Area's " S A D S, S A F S will be amongst themselves, as the " Times wherein they were described. Now let the Num- " ber of the Triangles be increased, and their Wideness di- " minished infinitely; both that last Perimeter of them, " A B C D E F, will be a curve Line, and the Area's de- " scribed in one and the same immoveable Plane, will in " this Case also be proportional to the Times as well as be- " fore. *Q. E. D.*

PROP.

PROP. XXXIX, XL. Theorems.

EQUAL Triangles (ACB, AFB) upon the same, or an equal Base (AB) and on the same Side, are between the same Parallels (AB, CF.) Fig. 64.

If you deny it, let CL be parallel to AB, and let BL be drawn. Then ALB is equal to ACB (*a.*) But by the Hypothesis, AFB is equal to ACB. Therefore ALB and AFB are equal; *i. e.* a Part is equal to the Whole. Which cannot be. Therefore, &c.

(a) *By the foregoing.*

[Corollary (1.) "Hence also, with the famous Sir " *Isaac Newton*, we gather, that all Bodies which are " moved in Curve Lines, and describe Area's about some " Center proportional to the Times, are perpetually urg'd " and press'd by a Force impelling towards the Center. " For because of the Equality of the Triangles SCB, ScB " described upon the same Base SB, the Points C and c " shall be in a Line Cc, which is parallel to the Base; and " so the Figure BcCg shall be a Parallelogram; the Sides " whereof Bc and Bg are * the Lines of the Directions of " the Forces, and BC is the Diagonal. The Body there- " fore is urged unto C by the Force Bg, which tends unto " S the Center. And so in all the Points, C, D, E, F. " *Q. E. D.*

* *Per Corol.* 2. *Prop.* 33. *l.* 1.

Corollary (2.) "Seeing therefore in the Motion of " the primary Planets, the Area's made by Rays, or right " Lines drawn from them unto the Sun, are always pro- " portional to the Times, as all Astronomers know, the " Planets are urged by a perpetual Force, which tends to " the Sun. And the same thing is equally true of the se- " condary Planets, with respect to their primary ones.]

PROP. XLI. Theorem

Fig. 65. *IF a Triangle (AFB) be in the same Parallels with a Parallelogram (AL) and have the same, or an equal Base (AB) it is half of the Parallelogram.*

(a) *Per* 37, 38. *l.* 1. (b) *Per* 34. *l.* 1.

Draw C B. The Triangles A F B, A C B are (*a*) equal. But A C B is half of the Parallelogram A L (*b*.) Therefore A F B also is half of A L. *Q. E. D.*

Scholium.

Fig. 65. FROM this Propoſition, with the Scholium of *Prop.* 35. we learn, that the Area of whatſoever Triangle, as A F B, is produced from half the Altitude F I multiplied into the Baſe A B, or half the Baſe multiplied into the Altitude. Wherefore one Side of a Triangle being known, and the Height, that is, the Perpendicular which falls upon the known Side from the oppoſite Angle, the Meaſure of the Triangle is given. As if the Baſe AB be of an 100 Feet, the Height F I, 85. multiply half the Baſe, 50 by 85, and you have the Area of the Triangle A F B=4250 Feet Square. Further, the Altitude of a Triangle, when the Area of it is in all Points acceſſible, may be known mechanically as well as the Sides. But if the Area of it cannot be gone over, the Height may be found Geometrically by 12 and 13, *Lib.* 2. as we ſhall there ſhew.

In a Rectangle Triangle, the Height is the ſame with either of the Sides about the right Angle. Half of this therefore multiplied into the other Side adjacent to the right Angle, will give the Area of the Triangle.

PROP. XLII. Problem.

Fig. 66. *TO make a Parallelogram with an Angle equal to a given one (O,) and equal to a given Triangle (ACB.)*

(a) *Per* 31. *l.* 1. (b) *Per* 23. *l.* 1. (c) *Per* 31. *l.* 1.

Biſect the Baſe A B in F. Through C draw C X parallel (*a*) to A B. Make the Angle B A L equal to the given one O (*b*.) Draw F I parallel (*c*) to A L. A L, I F ſhall be that which was ſought for.

For

For let F C be drawn. The Parallelogram A I hath an Angle LAF equal to the given one O, and is equal to the given Triangle A C B; ſeeing that, as well the Triangle A C B *(d)* as the Parallelogram A I *(e)* is double to the ſame Triangle A C F.

(d) *Per* 38. *l.* 1.
(e) *By the foregoing.*

Corollary.

THE Triangle A C B being given, a Rectangle equal to it is had, if there be drawn a Line parallel to the Side A B, and A B being biſected in F, the Perpendicular B Q be erected. For the Rectangle under F B and Q B will be equal to the Triangle A C B. *Fig.* 66.

PROP. XLIII. Theorem.

IN a Parallelogram (as B L) the Complements (B O, O L) of thoſe Parallelograms which are about the Diameter (R F, C S) are equal. *Fig.* 67.

If through any Point of the Diameter A Q, as the Point O, C F be drawn parallel to the Side A B, and R S parallel to the Side B Q; the whole Parallelogram B L is divided into four Parallelograms, whereof two are about the Diameter R F, C S, the other two, B O, O L, are the Complements of theſe unto the whole Parallelogram B L.

Their Equality is thus proved. The Triangles *(f)* A B Q, A L Q are equal. Likewiſe the Triangles A R O, O C Q *(g)* are equal to the Triangles A F O, O S Q. Therefore, if from the Equals *(h)* A B Q, A L Q, you take away Equals, on this Side A R O, O C Q on that A F O, O S Q; then B O and O L ſhall remain equal. *Q. E. D.*

(f) *Per* 34. *l.* 1.
(g) *By the ſame.*
(h) *Per Axiom* 3.

PROP. XLIV. Problem.

UPON a given right Line (O S) to conſtitute a Parallelogram, in a given Angle (X,) which Parallelogram ſhall be equal to a given Triangle (V.) *Fig.* 68.

Make a Parallelogram *(a)* R C equal to the given V, having its Angle R O C, equal to the given one X, and

(a) *Per* 42. *l.* 1.

join the Side R O directly to the given Line O S, so as to make one right Line therewith. Then through S draw
(b) *Per* 31. *l.* 1. S Q (*b*) parallel to O C, which S Q, let B C meet when it is produced unto Q. Then let a right Line, drawn through Q and O, meet B R produced unto A. Which done, through A draw A L, parallel to O S, which A L, let C O and Q S meet, when it is produced unto F and L; the Parallelogram O L is that which was required.

(c) *By the foregoing.* (d) *Per* 15. *l.* 1. For O L (*c*) is equal to R C, that is, by the Construction, to the given Triangle V, and is at the given Line O S; and (*d*) the Angle F O S is equal to the Angle R O C; that is, by the Construction, equal to the given Angle X.

Scholium. "This Proposition contains a certain Geometrical Division. For in the vulgar Arithmetical Division, the Number to be divided may justly be considered
Fig. 91. "as being a certain Rectangle: *e g.* Let the Rectangle "A B comprehending 12 Square Feet, be to be divided by "2; *i. e.* a Rectangle is to be found equal to that A B "of 12 Square Feet, one of whose Sides shall be only 2 "Feet: From whence it comes to be enquired of what "Number the Side sought shall consist; which Side is to "be esteemed a certain Quotient of this Division. Which "thing is performed Geometrically after this manner. "With a pair of Compasses take the Line B D of 2 Feet, "and draw the Diagonal D E F. The Line A F is that "which is sought for. For the Complements E G and
* *Per.* 42. *l.* 1. "E C are * equal; and in the Rectangle E G, one Side, "E H, is equal to the Line B D, which is of 2 Feet; "and the Side E I, is equal to A F.

"This Kind of Division is called *Application*, because the "Rectangular Space A B is *applied* to the Line B D or "E H; and hence it comes, that Division is often named "*Application*; respect being had to the Practice of the old "Geometricians, who always made more Use of Geometrical Construction, which requires only a Rule and a "Pair of Compasses, than of Arithmetical Computation, "which is performed by Numbers.

PROP.

PROP. XLV. Problem.

UPON a given Line (I Q) and in a given Angle (H) to make a Parallelogram equal to a given Rectilineal Figure (C B A.) *Fig.* 69.

Resolve the given Rectilinear into the Triangles A, B, C, by drawing the right Lines F L, F I.

Upon the given Line I Q. in the given Angle H, make (*a*) the Parallelogram I V equal to the Triangle A. Then the right Line I R being produced infinitely towards P; upon the right Line R V, in the Angle V R P, (*b*) make the Parallelogram R Z equal to the Triangle B. Again, upon the Line S Z, with the Angle Z S P, make the Parallelogram S G equal to the Triangle C. This done, I say, I G is the Parallelogram sought for. (a) *Per* 44. *l.* 1. (b) *By the same.*

For (*c*) the Angle Z V R is equal to its Alternate I R V. But (*d*) Q V R and I R V, are equal to two right Angles. Therefore also Q V R and Z V R, are equal to two right ones. Therefore * Q V and Z V fall directly so as to make one right Line. After the same manner I might shew that Q Z and Z G make one right Line. Therefore the whole Q V Z G is one right Line, and is also parallel to I X, seeing by the Construction Q V is parallel to I P. Now X G also (*e*) is parallel to I Q. Seeing X G is parallel to S Z, and S Z to R V, and R V to I Q. (c) *Per* 27. *l.* 1. (d) *By the same.* * *Per* 14. *l.* 1. (e) *Per* 30. *l.* 1.

I G therefore (*f*) is a Parallelogram; but that it is such an one as was required, is manifest from the Construction. (f) *Per Def.* 35.

[Corollary. "Hence is easily found the Excess whereby " a greater Rectilinear Figure exceeds a lesser: To wit, if " unto the same right Line I Q be applied Parallelograms " respectively equal to the two right-lin'd Figures. For " that Parallelogram, by which the greater Rectilinear ex- " ceeds the lesser, will give the Difference of them. *Q. E. I.*]

Scholium.

WE will here add a Problem that will be useful for the Practice of Proposition 14. *l.* 2.

A Quadrangular Figure, B F, being given, to describe an equal Rectangle. *Fig.* 70.

Resolve

Resolve it into Triangles by the right Line A C. From the opposite Angles, let down the Perpendiculars B O, F I. Bisect A C in S. From S erect the Perpendicular S L, equal to the two, B O, F I, put together. The Rectangle, comprehended under L S and S C, is equal to the given, B F. The Demonstration appears out of Proposition 41.

PROP. XLVI. Problem.

Fig. 71. *FROM a given right Line (A B) to describe a Square.*

Erect two Perpendiculars equal to the given A B; to wit, A C, B E, then join C E. I say, the Thing is done. For seeing the two Angles A and B are (g) right ones, A C and B E shall (h) be parallel; but they are also (a) equal. Therefore C E and A B are (b) parallel and equal. Therefore the Figure is Parallelogram and Equilateral. But all the Angles also are right ones (for seeing A and B are right Angles, the opposite ones (c) E and C are right also.) Therefore the Figure A E is a Square.

(g) *By the Construction.*
(h) *Per* 29. *l.* 1.
(a) *By the Construction.*
(b) *Per* 33. *l.* 1.
(c) *Per* 34. *l.* 1.

[" In the same manner you may easily describe a Rect-" angle, which hath the two unequal Sides given.]

PROP. XLVII. Theorem.

IN every Right-angled Triangle (as A B C) the Square of the Side (A C) which is opposite to the right Angle, is equal to the two Squares together of the two other Sides (A B, C B.) Fig. 72.

Let I C and B F be drawn; and B E parallel to A F. Now, if to the right, and therefore equal Angles I A B, F A C, there be added the common Angle B A C, the Wholes, I A C and F A B, shall be equal. But in the Triangles, I A C, F A B, the Sides which contain those equal Angles, are equal (d) amongst themselves, to wit, I A, C A, to B A, A F, each to each. Therefore the Triangles, I A C, F A B, (e) are equal. Which, because they stand upon

(d) *Per Def. Square*
(e) *Per* 4. *l.* 1.

upon the ſame Baſes, I A, F A, with the Parallelograms, A B L I and Z A F E, and between the ſame Parallels, I A, L B C, and A F, E Z B, they are halves (*f*) of thoſe Parallelograms. Therefore the Parallelograms, A B L I, Z A F E, as being Doubles of Equals, are equal betwixt themſelves. By the ſame reaſoning, if right Lines, A X, B R, were drawn, it might be ſhewn that the Parallelograms E C, B X, are equal. Therefore the whole, A R is equal to I B and B X, together. *Q. E. D.*

(f) *Per* 31. *l.* 1.

It was taken for granted that L B C is parallel to I A, in order to which L B and B C muſt be one right Line. Now that they are ſo, is manifeſt from the 14th, ſeeing the Angles L B A and C B A, are both right ones by the Hypotheſis.

Scholium.

THIS Theorem (which, *Prop.* 31. *l.* 6, *Euclid* extends unto all like or ſimilar Figures) is commonly call'd the *Pythagoric* Theorem, from *Pythagoras*, the Inventor of it; who, as is atteſted by *Proclus*, *Vitruvius* and others, offer'd Sacrifices to the Muſes, as ſuppoſing himſelf to have been helped by them in ſo excellent an Invention; in which thing he ſhew'd himſelf to be ignorant of God, the Lord of Sciences, the true and only Author of all Wiſdom; or certainly, if he knew him, he glorified him not as God. There is frequent and notable Uſe of this Theorem through all the Mathematicks; and in particular, it opens a Way unto the Knowledge of incommenſurable Magnitudes, a main Secret of Geometrical Philoſophy.

That the Side of a Square is incommenſurable to the Diameter, is a Thing much celebrated amongſt the old Philoſophers, *Ariſtotle* and *Plato* eſpecially; inſomuch that *Plato* would ſay, that he who knows not this, is not a Man, but a Beaſt. Now the Knowledge of this Myſtery ſeems to to have taken its Riſe out of this 47th Propoſition. For ſeeing in the Square A E, the Angle A is a right Angle, the Square of the Diameter C B ſhall be equal to both the Squares of the Sides, A B, A C, and therefore double to one of them. Wherefore ſeeing the Square of C B is 2, and the Square of the Side A B is 1, or Unity, the Diameter C B ſhall be the Square Root of 2, and the Side A B the Square Root of Unity, it ſelf; the *Ratio* of which Quantities

Fig. 71.

Quantities (as it will be demonſtrated in its Place) cannot be explicated in Numbers, and therefore they are incommenſurable.

And by this one Argument alone, if all others were wanting, it might evidently be made out, that Geometrical Magnitudes cannot be made up of a definite Number of Points: for otherwiſe none would be incommenſurable; foraſmuch as a Point would be the common Meaſure of all.

To theſe Things we will ſubjoin three Problems, which are deduced out of the preſent Propoſition, and are of frequent Uſe.

Problem 1.

Fig. 73. IF any Number of Squares are given, to make one equal to them altogether.

Let there be three or more Squares given, whoſe Sides are A B, B C, C E. Make the right Angle F B Z, having indefinite Sides, and unto the Sides of it transfer A B and B C, and then join A C. The Square of A C ſhall be equal to the Squares of A B and B C together (*a*.) Then transfer A C from B unto X, and C E the third given Side, transfer from B unto E, and join E X; the Square of E X ſhall be equal (*b*) to the Squares of E B (or E C) and B X together; that is, equal to the three given Squares, whoſe Sides are AB, BC, CE: And ſo on as long as you pleaſe.

(a) *Per* 47. *l.* 1.

(b) *By the ſame.*

Problem 2.

Fig. 74. TWO unequal right Lines being given (A B, B C) to determine that Square, whereby the Square of the greater (A B) exceeds the Square of the leſs (B C.)

From the Center B, with the Interval A B, deſcribe a Circle. Then from C erect a Perpendicular C E, cutting the Circumference in E. The Square of C E is the Exceſs or Difference which is ſought for.

For let E B be drawn. The Square of B E, that is, of A B is equal to the Squares (*a*) of B C and C E together. Therefore, *&c.*

(a) *Per* 47. *l.* 1.

Problem 3.

Fig. 75. ANY two Sides of a Right-angled Triangle being known, to find the third.

Let

Let the Sides containing the right Angle be A B, A C, the one of 6 Feet the other of 8. You are to find of how many Feet the Side C B, which is oppofite to the right Angle, is. To do which, multiply 6 and 8 each of them by it felf. From which Multiplication there will arife for the Squares of thofe two Sides 36 and 64; the Sum of which is 100. The fquare Root of 100, which is 10, gives the Feet of the Side B C, whofe Quantity was fought. This Demonftration offers it felf in and from this 47th Propofition, for the Sum of the Squares B A and C A is equal to the Square of B C. Therefore the Root of the Sum of them is equal to the Root or Side B C.

Then let the Sides A B, B C be known, the one of 6 Feet, the other of 10, you are now to find AC. Take the Square of the Side AB which is 36, out of the Square of the Side BC=100. The Remainder 64 fhall be the Square of the Side A C. The Root therefore of 64, which is 8, gives the Feet of the Side A C.

Corollary. " From hence we derive the Original of *Fig.* 92.
" the Tables of Sines, Tangents and Secants. For In-
" ftance, let A C the Semi diameter of the Circle be of
" 100,000 Parts, and the Angle B A D of 30 Degrees.
" Becaufe the Chord or Subtenfe of 60 Degrees is * equal *Per Corol.*
" to A C the Semi-diameter; B D the Sine of 30 Degrees 1. *Prop.* 15.
" fhall be equal to half the Semidiameter, or $\frac{1}{2}$ A C; and *l.* 4. *and Co-*
" therefore fhall contain 50,000 Parts. But now in the *rol.* 2. *Prop.*
" right-angled Triangle A D B, the Square of A B is equal 3. *l.* 3.
" to the Squares of A D and B D. Wherefore let the
" Semidiameter A B be fquared (by multiplying 100,000
" by 100000) and from that Square fubtract the Square of
" B D. The Remainder fhall be the Square of A D, or
" of the Cofine equal to it B F; out of which extract the
" fquare Root, and you will have the Line B F or A D.
" Then by this following Analogy, A B : B D : : A E :
" C E, or A D : B D : : A C : C E, will be had the Tan-
" gent C E. And then laftly, if the Square of A C be
" added to the Square of C E, the Root of the Sum being
" extracted will be the Secant A E. *Q. E. I.*

PROP.

PROP. XLVIII. Theorem.

Fig. 76. *IF in a Triangle the Square of one of the Sides (AB) be equal to the two Squares of the other Sides (AC, BC) taken together, the Angle (ACB) which the two other Sides contain, is a right Angle.*

If not, the Angle A C B will be greater or leſs than a right Angle. In either of which Caſes (as it will be demonſtrated, *Prop.* 12, 13. *l* 2. which Propoſitions depend not on this) the Square of A B will not be equal to the Squares of A C, B C together; which is contrary to the Hypotheſis.

Or thus. Draw F C perpendicular to A C, and equal to
(a) *Per* 47. C B, and join A F. The Square of A F is (*a*) equal to the
l. 1. Squares of F C, C A together; that is, (*b*) to the Squares
(b) *By the* of B C, C A; that is, by the Hypotheſis, to the Square
Conſtruction. of A B. Therefore the right Lines A F, A B are equal.
Becauſe therefore the Triangles X and Z are mutually
(c) *Per* 8. equilateral, the Angles at C (*c*) are equal. Therefore
l. 1. they are both right Angles (*d*). *Q. E. D.*
(d) *Per*
Def. 14.

THE

THE

Elements of EUCLID.

BOOK II.

THIS Book is small in Bulk, but great indeed in the Excellence and Usefulness of its Theorems. Young Beginners will not, I know what I say, be at first able to discover it; but being further advanced, they will, from their own Experience, and with the greatest Certainty, apprehend that it is most true.

A DEFINITION.

A Right-angled Parallelogram (as A E) which is wont simply, and without any Addition, to be call'd a Rectangle) is said to be contain'd under the two Lines (AC, A F) which determine the Magnitude of it. *Fig.* 60. *l.* 1.

For the one of them A C determines the Heighth, the other A F the Breadth of it. Now, if the Side A C be understood to be carried perpendicularly along the whole A F, or A F along A C, by that Motion the Rectangle or its Area will be produced. Wherefore a Rectangle is rightly said to be produced from the drawing of two Lines into one another, or the Multiplication of them one by the other. When therefore you have these Words, [the Rectangle under (or of) A C, C B,] or for Brevity's sake, [the Rectangle A C B,] there is meant that Rectangle which is contained under A C and C B, multiply'd one into the other. In like manner, when we say the Rectangle under A B, B C, or the Rectangle A B C, there is designed the Rectangle contained *Fig.* 2. *l.* 2.

tain'd under the right Lines A B and B C, multiply'd by one another.

Moreover, of Rectangles ſome are Oblong, ſome are Square. The Oblong Rectangle is that which hath its contiguous Sides unequal, or which is contained under two unequal right Lines. The ſquare Rectangle that which is contain'd under two equal right Lines.

PROPOSITION I. Theorem.

Fig. 1. *l.* 2. *IF there be two right Lines (AB, AC), one whereof is divided into as many Parts as you will (AE, EF, FC;) the Rectangle compriz'd under thoſe two (AB, AC) is equal to all the Rectangles together, which are contain'd under the undivided Line (AB) and the ſeveral Parts of the divided Line (AE, EF, FC).*

Make A B perpendicular to A C, thro' B draw the infinite Line B R parallel to A C. From E, F, C, erect the Perpendiculars E I, F L, C Q. B C will be a Rectangle under A B and A C; and, is equal to the Rectangles B E, I F, L C; that is, (becauſe as well I E as L F are equal * to A B) equal to the Rectangles under A B, A E; A B, E F; A B, C F.

* *Per* 29, & 34. *l.* 1.

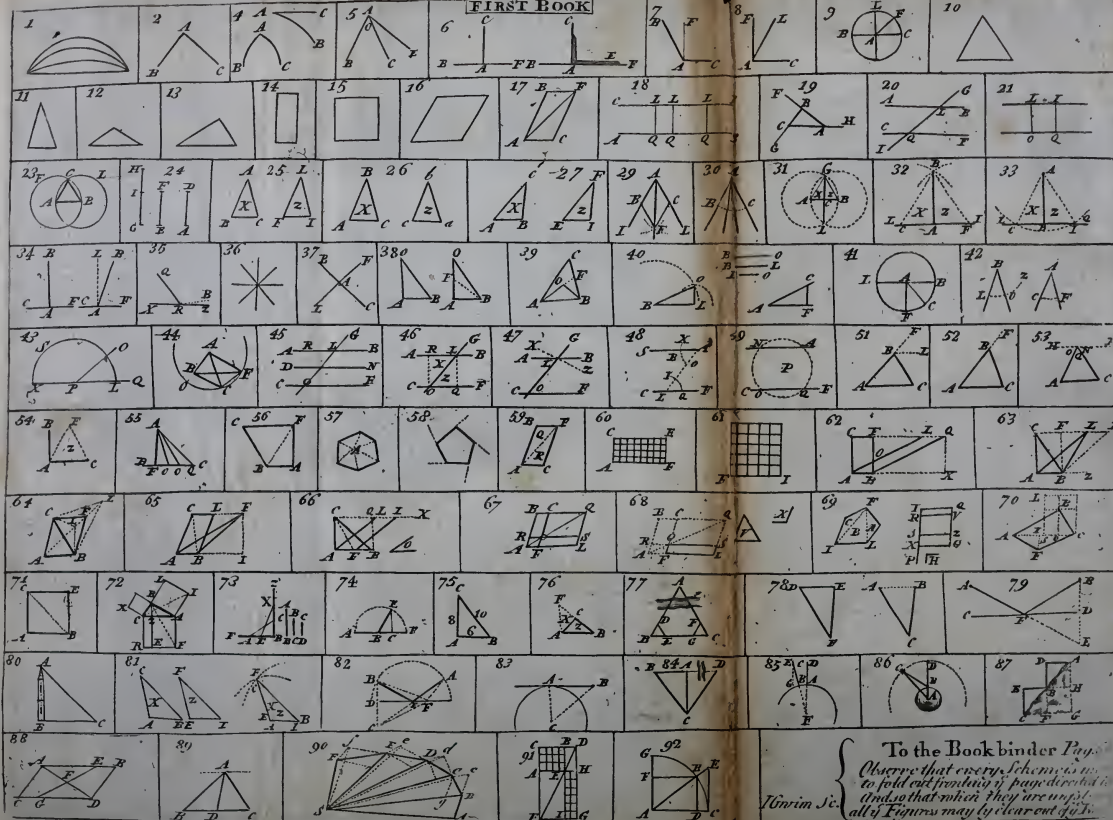
Scholium.

THE ten firſt Theorems of this Book are true alſo in Numbers, if they as Lines be divided into Parts. The numerical Rectangles are produced from the Multiplication of two Numbers, and the numerical Squares from the Multiplication of the ſame Number by itſelf.

[" Let the undivided Number be 9, and the divided one " 12. The Rectangle which is from 9 multiplying 12=108 " will be equal to the three Rectangles, 27, 36, and 45, " which are produced from 9 multiplied by 3, and 4 and 5, " reſpectively and ſeparately. Or let the Number 432 be " as it were a Multiplicand divided into 400 and 30 and 2 " and the Number 8 an undivided Multiplier; 8×432= " 3456 will be equal to 8×400=3200+8×30=240+8×2 " =16. And from this Propoſition therefore the Demon- " ſtration of Multiplication is to be deriv'd.]

PROP.

FIRST BOOK



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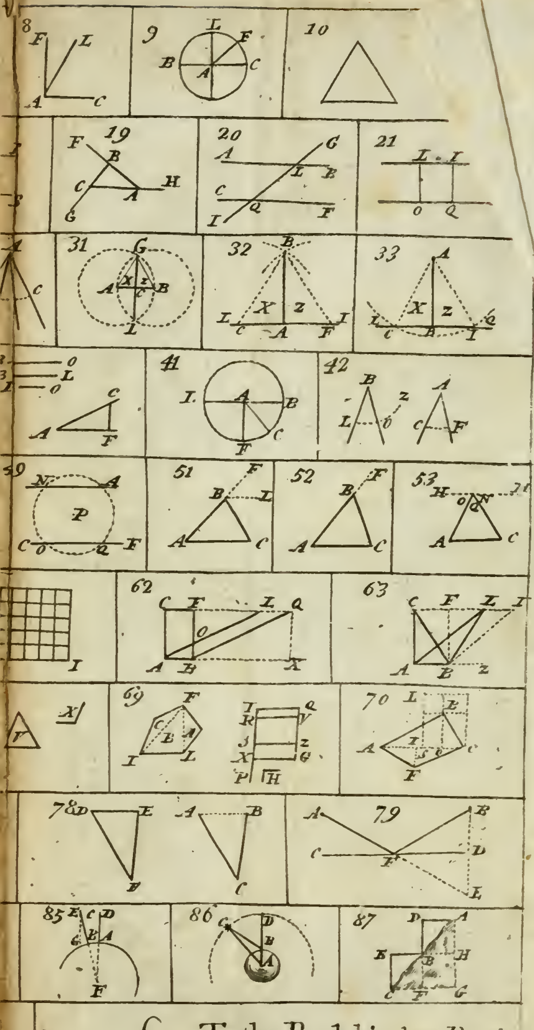

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of Book XI, XII. opposite Page 200

of Book XII. Continued Page 252

And last Plate at the End of the APPENDIX.

PROP. II. Theorem.

IF the right Line (AB) be cut any where (as in C,) the two Rectangles under the whole (AB) and the Parts (AC, CB) are equal to the Square of the whole Line (AB.) Fig. 2.

[" For AD is the Square of the whole, and AH, CD are Rectangles under the Whole AB, and the Parts AC, CB.] Fig. 17.

[" Let the Number 8 be divided into 5 and 3; the Square of the Whole $8 \times 8 = 64$, is equal to the Rectangles $8 \times 3 = 24$, $+ 8 \times 5 = 40$.]

PROP. III. Theorem.

LET a right Line, as (AB) be cut any where, (as for instance in C,) the Rectangle contain'd under the Whole AB, and either of the Parts, (BC) is equal to the Rectangle under the Parts (AC, CB) together with the Square of the said Part (BC.) Fig. 3.

[" For AF is the Rectangle under the whole Line AB, and the Part AC; and CF is the Rectangle under the Parts, as AE is the Square of the Part AC.] Fig. 18.

" In Numbers. Let the Number 7 be divided into the Parts 3 and 4. The Rectangle of $7 \times 3 = 21$ is equal to the Rectangle of $3 \times 4 = 12$, together with the Square $3 \times 3 = 9$. In like manner $7 \times 4 = 28$, is equal to the Rectangle $3 \times 4 = 12 +$, the Square $4 \times 4 = 16$.]

PROP. IV. Theorem.

LET a right Line, as (FL) be cut any where, as in (O,) the Square of the Whole shall be equal to the Squares of the Parts (FO, OL) and to two Rectangles contain'd under the Parts (FO, OL.) Fig. 4.

[" For FD is the Square of the Whole, and CG and CL the Square of the Parts; and CF, CD, two Rectangles under the Parts.] Fig. 19.

" In Numbers. Let the Number 10 be divided into " two Parts, 7 and 3. The Square of $10\times10=100$ is " equal to the Squares of the Parts $7\times7=49$, and 3×3 " $=9$, and to the two Rectangles $7\times3=21$, and $7\times3=21$. " And on this Propofition depends the Extraction of the " Square Root.

Fig. 19. Corollary (1.) " Hence it is manifeft, that the Parallelo- " grams about the Diameter of a Square, (O I, H K) are " Squares.

(2.) " As likewife, that the Diameter of every Square " bifects the Angles of it.

(3.) " And that the Square of half the Line is a fourth " Part of the Square of the whole Line. For in that Cafe " the Rectangles and Squares end in four equal Squares.

PROP. V. Theorem.

Fig. 5. *IF a right Line, as (QX) be cut equally in (R) and unequally in (S,) the Rectangle contain'd under the unequal Parts (QS, SX) taken together with the Square of the intermediate Part (RS) ſhall be equal to the Square of the half (QR.)*

Fig. 20. [" For QH is the Rectangle under the unequal Parts, " and L G the Square of the intermediate Part, and R F " the Square of half the Line; and therefore, becaufe the " Rectangle QL is equal to the Rectangle S F, and the reft " of the Space is common to both, the Propofition is mani- " feft.]

" Let the Number 8 be divided equally, that is, into " 4 and 4, and unequally into 5 and 3. The Rectangle " of $5\times3=15$ together with the Square $1\times1=1$ ſhall be " equal to the Square $4\times4=16$.]

PROP. VI. Theorem.

Fig. 6. *IF a right Line (AB) be divided into two equal Parts in C, and to it a certain right Line (BF) be adjoin'd; the Rectangle contain'd under the whole compound Line (AF) and the adjoin'd one (BF) taken together with the Square*

of

of half the Line (CB) shall be equal to the Square of (CF) which is compounded of half the Line (AB) and the adjoin'd one.

[" For AN is the Rectangle under the whole compound Fig. 21.
" Line and the adjoin'd one; and GK the Square of half
" the Line AB; and CE the Square of the Line com-
" pounded of half the Line AB, and that which was ad-
" ded. Wherefore, because the Rectangle HE is equal to
" the Rectangle AK, and the rest of the Space is common
" to both, AN and KG is equal to CE. *Q. E. D.*]

[" If the Number 6 be divided into the two equal Parts,
" 3 and 3; and to it be added the Number 2; the Rect-
" angle of $8\times2=16$, taken together with the Square 3×3
" $=9$, shall be equal to the Square $5\times5=25$.]

Corollary. " Hence, with *Maurolicus,* with one single
" Observation, we learn to measure the Diameter of the
" Earth. Let the Altitude of the Mountain AD be known, Fig. 22.
" and AB the Line touching the Earth be known by
" measuring. Let the Line DE be cut into two equal
" Parts in the Center C, and to it be added the Line
" AD. Now, because the Rectangle under AE, AD, to-
" gether with the Square of DC, is by this Proposition
" equal to the Square of AC, that is, equal to the * Squares *Per 17. l. 1.
" of the Lines AB, BC: From hence it follows, that if
" you take away on both Sides the Square of CD or CB,
" the Rectangle which is under AE, AD is equal to the
" Square of AB. Therefore let the known Square of AB
" be divided by the known Altitude of the Mountain AD,
" and the Quotient will give the Line AE. From which
" subtract the known Altitude of the Mountain AD, the
" remaining Line DE will be the Diameter of the Earth.
Q. E. I.

PROP. VII. Theorem.

If a right Line (AB) be cut any where, (as in C,) the Square of the whole Line (AB) taken together with the Square of either of the Segments (AC) is equal to two Rectangles contained under the whole (AB) and that Segment (AC,) together with the Square of the other Segment (CB.) Fig. 7.

 [" For

Fig. 23. [" For E B is the Square of the whole Line, and A L " the Square of the Part A C. But the two Rectangles un- " der the whole Line, and that Part E I, H L, together " with G B, the Square of the other Part, possess the " same Space that E B and the Square of A C doth. " Therefore they are equal to E B and the Square of A C.

" Let the Number 13 be divided into any two Parts, " as 9 and 4. The Square 13×13=169, together with " that 9×9=81, is equal to 13×9=117, and 13×9=117, " and the Square 4×4=16.]

PROP. VIII. Theorem.

Fig. 8. *IF a right Line* (*LF*) *be divided into two equal Parts in* (*I*,) *and to it a certain right Line be adjoin'd* (*FO*;) *the Rectangle* (*LIO*,) *which is contain'd under the half of the Line* (*LI*) *and the Line* (*IO*) *that is compounded of half the aforesaid Line, and the Line adjoin'd, this Rectangle taken four times, together with the Square of the adjoin'd Line* (*FO*,) *shall be equal to the Square of the whole compound Line* (*LO*.)

Fig. 24. [" For A L is the Square of the whole Compound, con- " taining four equal Rectangles under L I and I O (to wit, " D R, B Q, R O, and the fourth made up of L R and " Q H added together) and with those four Rectangles the " Square H E. From whence the Proposition is manifest.

" Let the Number 12 be divided into 6 and 6; and the " Number 4 be added to it. The four Rectangles, 10×6 " =240 and 4×4=16 are equal to the Square 16×16=256.

PROP. IX. Theorem

Fig. 9. *IF a right Line* (*AC*) *be divided equally in* (*B*) *and unequally in* (*F*,) *the Squares of the unequal Parts* (*AF*, *FC*) *will be double to the Squares of half the Line* (*AB*,) *and of the intermediate Part* (*BF*.)

[" Let

[" Let B E be equal and perpendicular to B A. From " hence the Construction being made, as the Figure shews, " the Lines AB, BE, CB will be equal: As also the Lines " E G, G Q will be equal. The Angles A E C, A B E, " C B E, E G Q, Q F C will be right; and the Angles " A E B, B E C, E C A, C Q F, E Q G half right ones. " From whence the Square of A E will be double * to the " Square of A B, which is half of A C, and the Square of " E Q double to the Square of G Q or B F the intermediate " Line. But the Squares of A E and E Q are † equal to " the Square of A Q, that is, to the Squares of A F and " F Q or F C the unequal Parts. *Q. E. D.* * *Per* 47. *l.* 1. † *By the same.*

[" Let the Number 32 be divided equally into 16 and 16, " and unequally into 20 and 12. The Square 20×20=400, " with the Square 12×12=144, are double to the Squares " of 16×16=256 and 4×4=16.]

PROP. X. Theorem.

IF a right Line (FI) be divided into two equal Parts in (L,) and to it a certain right Line (as IO) be adjoin'd; the Square of the whole compound Line (FO,) taken together with the Square of the additional Line (IO,) shall be double to the Squares, which are described upon the half Line (FL,) and (LO) that which is compounded of half the Line (FI) and the additional Line. *Fig.* 10.

[" For a Construction being supposed not unlike to the " former; the Square of F E, is double to the Square of " the half Line F L, and the Square of E G is double to " the * Square of E Q or L O, which is compounded of " the half Line and the additional one. But the Squares " of F E and E G are equal to the Square F G; that is, " to the Square of F O, the whole compound Line, taken " together with the Square of O G or O I the additional " Line. *Q. E. D.* *Fig.* 26. * *Per* 47. *l.* 1.

[" Let the Number 40 be divided into 20 and 20, and " to it let there be added the Number 14. The Square " 54×54=2916, with the Square 14×14=196 are double " to the Square of 20×20=400, taken together with 34×34=1156.]

PROP. XI. Problem.

Fig. 11. *SO to cut the given right Line (AB) in (C) that the Rectangle (ABC) which is contain'd under the whole Line and one Part, ſhall be equal to the Square of the other Part (AC.)*

From A erect a perpendicular AF equal to AB. Biſect AF in X Draw the right Line XB; from the Line FA drawn forth, cut off XI equal to XB. Then cut off AC equal to AI. I ſay the Thing is done

For let the Square BAFS be perfected; and a Perpendicular being drawn through C, let the Rectangle FILO be perfected alſo. Becauſe FA is biſected in X, and to it is added AI; there ſhall be

(a) *Per* 6. *l.* 2.
(b) *By the Conſtruction.*
(c) *Per* 47. *l.* 1.

$$\left.\begin{array}{l}\text{the Rect. FIA}\\ \quad + \\ \text{Square of XA}\end{array}\right\} = (a)\text{ to the Square of XI}$$

That is, = to the Square of XB (b)

That is, = to the Squares of $\left.\begin{array}{r}\text{AB}\\ \text{AX}\end{array}\right\}$ (c)

Therefore let there be taken away on both Sides the Square of XA; there will remain

the Rectangle FIA or FL.

= AS the Square of the Line BA;

Wherefore again, the common Rectangle AO being taken away,

AL will remain equal to CS.

But AL is the Square of the Line AC, ſeeing by the Conſtruction AC and AI are equal. And CS is the Rectangle ABC, foraſmuch as BS is equal to AB. Therefore the Rectangle ABC is equal to the Square of AC. Therefore we have cut the Line AB, as it was required.

Scholium.

THE Ten firſt Propoſitions of this Book are true alſo in Numbers: But this Eleventh cannot be exemplify'd in Numbers; for no Number can be ſo divided that the Product of the whole multiplied by one Part ſhall be equal to the Square of the other. The Force of this Section of a Line is wonderful. For which, ſee *Prop.* 30. *Lib.* 6.

PROP. XII. Theorem.

IN an Obtuse-angled Triangle (ACB,) the Square of the Side (AB) opposite to the obtuse Angle (C,) exceeds the Squares of the other Sides (AC, CB,) by the Rectangle (BCF) twice taken; which same Rectangle is comprized under (BC,) one of the Sides containing the obtuse Angle, and the Line (CF) which is intercepted betwixt the Perpendicular (AF) and the obtuse Angle. Fig. 12.

The Square AB is equal to the Squares of AF, BF (*a.*) (a) *Per* 47. *l.* 1.

But the Square of BF is equal to the Squares of FC, CB, with the Rectangle FCB twice taken (*b.*) Therefore if you substitute these for the Square of BF; then the Square of AB is equal to AF Square, FC Square, CB Square, and Rectangle BCF twice. (b) *Per* 4. *l.* 2.

But the Squares of AF, FC are (*c*) equal to the Square of AC. Wherefore this being substituted for them, (c) *Per* 47. *l.* 1.

AB Square is equal to AC Square, CB Square + Rectangle BCF twice.

PROP. XIII. Theorem.

IN any Triangle whatsoever (as ACB) the Square of the Side (AB) opposite to an acute Angle (C) is exceeded by the Squares of the other Sides (AC, CB) by the Rectangle (BCF) twice taken; which same Rectangle is contain'd under (BC) one of the Sides comprehending the acute Angle (C;) and the Line (FC) which is intercepted betwixt the Perpendicular (AF) Fig. 13, 14.

let

let fall upon the Side (BC) from its opposite Angle (A) and the acute Angle (C.)

(d) *Per* 4. *l.* 2. The Square of B C is equal to (*d*) the Rectan. B F C (twice, + F C Square + F B Square

(e) *Per* 47. *l.* 1. And A C Square is equal to (*e*) C F Square + F A Square

Wherefore the two together B C Squ. A C Squ. are equal to Rect. B F C (twice + B F Square + 2 F C Square + A F Square

But the Rectangle B F C twice; together with the Square of F C twice, is (*a*) equal to the Rectangle B C F twice. Therefore this being substituted for them.

(a) *Per* 3. *l.* 2.

B C Squ. + A C Squ. are equal to the Rectang. B C F twice + B F Square + A F Square

(b) *Per* 47. *l.* 1. But the Squares of A F, B F are equal to (*b*) the Square of A B. Therefore this being substituted for them.

B C Squ. + A C Squ. are equal to the Rectangle B C F twice + A B Square

That is, B C Square + A C Square do exceed A B Square by the Rectangle B C F twice taken.

Corollary.

Fig. 15. THE Proposition is true, although the Perpendicular falleth without the Triangle. And the Demonstration is almost the same.

(c) *Per* 12 *l.* 2. [" More briefly thus. A C q = (*c*) A B q + C B Q + " 2 C B F. And on both Sides C B q, then A C + C B q

(d) *Per* 3. *l.* 2. " = A B q + 2 C B q + 2 C B F = (*d*) A B q + 2 B C F. " *Q. E. D.*

Scholium.

Scholium.

FROM this Propoſition, and the 47th of the former Book, we have the Meaſure of any Triangle whatſoever, whoſe three Sides are known, although the Area be altogether inacceſſible. For by the help of theſe Theorems, the Perpendicular is known, albeit the Impediments of the Place ſhould not permit us to mark it out. But Note, That the Perpendicular, multiplied by half the Side on which it falls, produceth the Area of the Triangle, as appears out of the *Scholium* of the 41ſt Propoſition, *Lib.* 1.

Let there be any Triangle (as A B C) having its Sides known. It is required to give the Perpendicular A F, which falls from the given Angle A upon the oppoſite Side C B. *Fig.* 15, *or* 14.

Take the Square of the Side A B oppoſite to the acute Angle C, out of the Sum of theSquares of AC, and BC. By the 13th, the Remainder ſhall be the Rectangle B C F twice taken. Divide half of the Remainder, that is, the Rectangle B C F by the known Side B C; thence will ariſe the right Line CF. Take the Square of the right Line CF out of the Square of A C. The Remainder will give (*a*) the Square of A F, whoſe ſquare Root will give the Perpendicular A F.

(a) *Per Prob.* 2. *Schol. poſt* 47. *lib.* 1.

This thing alſo may be obtained out of the 12th Propoſition. But the 13th ſufficeth, foraſmuch as in every Triangle the Perpendicular let fall from ſome one of the Angles unto the oppoſite Side, falls within the Triangles.

PROP. XIV. Problem.

THE Right-lin'd Figure (QXZ) being given, to make a Square equal to it. *Fig.* 68.

Make (*b*) a Rectangular Parallelogram C I equal to the Rectilinear QXZ; the Sides of which Parallelogram, if they ſhall be equal, you have already made the Square which was required; if they be unequal draw forth the greater Side I A unto L, until A L ſhall be equal to A C. Then biſect I L in Z; from which, as from a Center through

(b) *Per* 45. *l.* 1.

through I and L describe a Circle, and let C A be produced till it meets the Circumference in B. The Square of the right Line A B is equal to the given Rectangle Q X Z.

For let the right Line Z B be drawn; because I L is cut equally in Z, and unequally in A; the Rectangle

(c) *Per* 5. *l.* 2 $\left.\begin{array}{l}\text{I A L}\\ +\text{Z A Square}\end{array}\right\}$ are equal (*c*) to Z L Square, that is, equal

(d) *By the Construction.* to (*d*) Z B Square, that is, equal to (*e*) Z A Square + A B Square.

(e) *Per* 47. *l.* 1. Taking away therefore on both sides the common Z A *q*, there remains

Rect. I A L equal to A B *q*; that is,

Because A C and A L are equal, the Rect. C I equal to A B Square, and consequently A B Square equal to the

(g) *By the Construction.* Rectilinear (*g*) Q X Z:

Scholium.

EUCLID's Construction of this Problem requires that the given Rectilinear reduced unto a Rectangle by *Prop.* 45. *l.* 1. Which Reduction being operose enough, the Problem perhaps will more readily be dispatch'd after this manner.

Let the given Rectilinear be resolv'd into as many Quadrangles (X, Z) as it can. Then to each Quadrangle (*a*) make an equal Rectangle. If there remain, as here it happens, one Triangle (Q,) to it also (*b*) make a Rectangle equal. Then to each Rectangle by this 14th, *l.* 2. make an equal Square; and lastly, to all these Squares let one equal one be made (*c*.) This will be equal to the given Rectilinear Q X Z.

(a) *Per Schol. P.*45. *l.* 1.
(b) *Per Corol. p.*42. *l.* 1.
(c) *Per Prob.* 1. *Schol. p.*47. *l.* 1.

THE

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Elements of EUCLID.

BOOK III.

THE Fundamental Properties of the moſt perfect amongſt Plain Figures are demonſtrated in this Book. The Uſefulneſs of the Book is manifeſt by this one Thing alone, that it treats of a Circle, that abundant Source of admirable Things through the whole Mathematicks. The more famous Theorems are 16, 20, 21, 22, 31, 32, 35, 36.

DEFINITIONS.

1. THOSE Circles are equal, whoſe Diameters or Semi-diameters are equal. *Fig.* 20. *l.* 3.

2. A right Line (F B) is ſaid to touch a Circle, when it it doth ſo meet it in the Point (B,) that albeit it be produced, it doth not cut it.

3. Circles are ſaid to touch one another, when they do ſo meet that they do not cut each other. *Fig.* 13, 14.

4. In a Circle the right Lines (B C, F L) are ſaid to be equi-diſtant from the Center (A,) when the Perpendiculars which are let fall upon them from the Center (A O, A I) are equal. *Fig.* 18.

5. Segments or Portions of a Circle are the Parts into which the right Line (C E) which cuts the Circle doth divide it. *Fig.* 37.

6. An Angle in a Segment is that (B Q C) which is contain'd under the right Lines, which are drawn unto one Point of the Circumference (Q) from the Ends of the Segment (B C.) *Fig.* 33.

7. The

Fig. 33. 7. The Angle (C Q B) is said to stand upon the Circumference (BOC,) as being opposite to it.

Fig. 11. 8 A Sector is that Part of a Circle which is contain'd by two Semi-diameters, as (AB, A F) and an Arch as (B F or BCF) intercepted betwixt the Semi-diameters

PROPOSITION I. Problem.

Fig. 1. *l.* 3. *TO find the Center of a given Circle.*

Let the right Line (B C) be drawn in the Circle at random, which bisect in Q. Through Q draw the Perpendicular LF, which bisect in A. A shall be the Center.

If you deny it; let the Center be O, which is without the right Line FL (for in FL it cannot be, forasmuch as this Line is divided every where unequally but in A) and let there be drawn B O, Q O, C O. Because therefore you suppose O to be the Center, B O, C O must be equal; and the Triangles B O Q, C O Q must be equilateral to each other; seeing by the Construction B Q and C Q are equal, and Q O is common. Therefore the Angle O Q C (*a*) is equal to the Angle O Q B. Therefore O Q C is a right Angle (*b*) and consequently equal to L Q C, which is a right one by Construction, a Part to the Whole. Which is absurd.

(a) *Per* 8. *l.* 1.
(b) *Per Def.* 14. *l.* 1.

Corollary.

FROM what hath been demonstrated it appears, that if the right Line (L F) cuts another right Line B C into two equal Parts and perpendicularly, the Center is in the Line that cuts the other.

Fig. 2. The Center of a Circle is very easily found by a Square; the top of it (Q) being applied to the Circumference; for if the right Line D E, joining the Points D and E, in which the Sides of the Square cut the Circumference, be bisected in A, (A) shall be the Center. The Demonstration whereof depends on the 31st Proposition, *Lib.* 3.

PROP.

PROP. II. Teorem.

IF in the Circumference of a Circle there be taken two Points (C and B) the right Line which is drawn through them falls entirely within the Circle.

Let there be taken in the Line BC any Point whatfoever, as O, and from the Center A, be drawn A B, A O, A C. Becaufe A B, A C are equal, the Angles alfo B and C are (c) equal. Becaufe therefore A O C is (d) greater than the internal one B, it fhall be greater alfo than C. In the Triangle therefore O A C, the Side A C fubtending the greater Angle A O C, is (e) greater than the Side A O, fubtending the leffer Angle C. Seeing therefore AC reaches no farther than from the Center to the Circumference, A O fhall not reach fo far. Therefore the Point O fhall fall within the Circle. The fame thing may be fhew'd of any other Point of the Line B C. Therefore B C falls wholly within the Circle. *Fig.* 2. (c) *Per* 5. *l.* 1. (b) *Per Corol* 1. *Prop.* 32. *l.* 1. (e) *Per* 19. *l.* 1.

The Propofition is alfo manifeft from the very Notion of a right Line and a Circle.

Coroll. " Hence it follows, that a right Line touching a
" Circle, toucheth it in one fingle Point only. For if it
" touched the Circumference in two Points, it would be a
" right Line drawn thro' two Points of the Circle, and
" confequently would fall within the Circle, contrary to
" the Definition of a Tangent. And by the like reafoning
" (in paffing from Planes to Solids) it might be prov'd, that
" every Plane toucheth a Sphere only in one Point.

PROP. III. Theorem.

IF in a Circle a right Line (B L) drawn thro' the Center bifects another (C F) not drawn through the Center, it will cut it perpendicularly. And if it cut it perpendicularly, it will bifect it. *Fig.* 3.

Part I. From the Center (A) let there be drawn A C, A F. The Triangles X and Z are Equilateral to each other.

other. For C O, F O are by the Hypothesis equal, and A C, A F are so, because drawn from the Center; while A O is common to both. Therefore the Angles A O C,
(a) *Per* 8. *l.* 1. A O F are (*a*) equal. Therefore right (*b*) ones. Which was the first Part.

(b) *Per* *Def.* 14. *l.* 1. (c) *Per* 47. *l.* 1. Part II. Because by the Hypothesis A O C, A O F are equal Angles; A C Square shall (*c*) be equal to the Squares of A O, O C together; and A F Square equal to the Squares of A O, O F together. Seeing therefore the Squares of A C, A F are equal, the Squares of A O, O C together, will be also equal to the Squares of AO, O F together: Wherefore taking away the common Square AO, the Squares of O C, O F remain equal. And therefore the right Lines O C, O F are equal. Which was the other Part.

Corollary (1.) "Hence in every equilateral Triangle, "and in that also which is only an *Isosceles*, a Line which "falling from the top of the Angle, bisects the Base, is "perpendicular to it. And on the contrary, a Line which "falling from the top of the Angle is perpendicular to the "Base, doth bisect it.

(2.) "Hence it follows, that half of the Chord of every "Arch, is the right Sine of half the Arch.

PROP. IV. Theorem.

Fig. 4, 5. *IF in a Circle two right Lines*, (*B C, F L*) *not drawn both of them through the Center, cut each other, they cannot bisect each the other.*

Fig. 5. For if one of them L F passeth through the Center, it is manifest that it shall not be bisected by B C which doth not pass through the Center.

Fig. 4. If neither of them passes through the Center, from the Center A draw A O. If now B C, F L were both bisected
(a) *By the foregoing* in O, the Angles AO C, A O L would (*a*) be right Angles, and consequently equal; the Whole to a Part, which is absurd.

PROP. V, VI. Theorems.

Fig. 6, 7. *CIrcles cutting each other, or inwardly touching one the other, have not the same Center.*

For

For if it were otherwise, the right Lines A B, A C F, drawn from the common Center A, would be equal; and A C would be equal to A F; a Part to the Whole, because they are both equal to A B. Which is absurd.

PROP. VII. Theorem.

If in a Circle there be taken any Point besides the Center (A,) as the Point (C,) and divers right Lines fall from thence unto the Circumference (as CB, CL, CO, CF;) Fig. 8.

1. *(CB) which passeth through the Center, will be the greatest.*

2. *The remaining Part of the Diameter (CF) will be the least.*

3. *Of the rest that will be the greater, which is nearer to the greatest.*

4. *And no more than two equal Lines can be drawn from the said Point (C,) which is different from the Center, unto the Circumference.*

Part I. Let A L be drawn from the Center A: Because A L, A B are equal, the common Line A C being added to each. A C and A L together are equal to C B. But A L + A C are greater than L C (*b.*) Therefore C B is greater than L C. In the same manner B C will be shew'd to be greater than any other. (b) *Per* 20. *l.* 1.

Part II. From the Center A draw A O. A O (that is, A F) is less than AC, CO (*c.*) Therefore taking away the common Line A C, C O remains greater than C F. In the same manner C F is prov'd to be less than C Q or any other. (c) *By the same.*

Part III. In the Triangles C O A, C L A, the Sides L A, AC, are equal to A O, A C, each to each. But the Angle L A C is greater than the Angle O A C. Therefore (*d*) the Base L C is greater than the Base O C. (d) *Per* 24. *l.* 1.

Part IV. This is manifest from what goes before. For if there could be three drawn equal, C O, C I, C Q, there would be two on the same Side equal: Which is contrary to Part III.

Corollary.

Corollary. " By the foregoing reasoning *Theodosius* gathered, that of the Arches of great Circles drawn upon " the Surface of a Sphere, from any Point diverse from " the Pole of a certain Circle unto that Circle, the greatest " is that which passeth thro' the Pole of that Circle; the " least, that which is drawn unto the opposite Point; and " of the rest, that is the greater, which is nearest to the " greatest; as also that no more than two equal Arches " can be drawn from that Point unto the Circle. And in " like manner may the Reader reason of himself on some " other of the Propositions of this Book; it being very " easy to pass from Planes to Solids in these Argumentations.

PROP. VIII. Theorem.

Fig. 9, 10. *If from a Point (A) taken without a Circle, there be drawn unto the Circle the right Line (AB, AC, AF,) or (AO, AQ, AR;*

1. Of those which fall upon the concave Circumference, the greatest is (AB) which passes through the Center (Z.)

2. Of the rest, that is the greater, which is nearer to the greatest (AB.)

3. Of those which fall without the Circle, or upon the convex Periphery, the least is (AO) which being produced would pass through the Center (Z.)

4. Of the rest, that which is nearer to the least is less than that which is farther off.

5. No more than two equal Lines can be drawn unto the Circumference from the same Point (A,) whether they fall within the Circle, or only without.

Fig. 9. Part I. From Center Z draw Z C; because Z C, Z B are equal, the common A Z being added to each, A Z+Z C are equal to A B. But A Z+Z C are (a) greater than A C. Therefore A B is greater than A C. In like manner A B will be shewed to be greater than any other whatsoever.

(a) Per 20. l. 1.

Part II. Draw Z F. Because in the Triangles A Z C A Z F, the Sides A Z, Z C are equal to A Z, Z F, eac

to each; but the Angle A Z C is greater than A Z F, therefore the Base AC (*b*) will be greater than the Base AF. (b) *Per* 24. *l.* 1.

Part III. Draw Z Q. The two Lines A Q, Q Z are greater than A Z (*c*.) Taking away therefore the Equals Z Q, Z O, there remains A Q greater than A O. In the same manner A O is prov'd less than any other. *Fig.* 10. (c) *Per* 20. *l.* 1.

Part IV. Draw Z R. The right Lines A Q, Q Z are less than A R, R Z (*d*); therefore the Equals Z Q, Z R being taken away, A R remains greater than A Q. (d) *Per* 21. *l.* 1.

Part V. This is manifest from the Four foregoing.

PROP. IX. Theorem.

IF from some Point within a Circle (as A) more than two equal right Lines can be drawn unto the Circumference; that Point is the Center. *Fig.* 11.

This is manifest from Part IV. of the 7th Proposition.

PROP. X. Theorem.

Circles cut each other in two Points only. *Fig.* 12.

For let them cut, if it may be, in more (B, C, F,) From A, the Center of the Circle L Q, let there be drawn to the Points B, C, F, the Lines A B, A C, A F; these will be equal. Because therefore from the Point A, within the Circle O S, there are drawn three equal Lines, A B, A C, A F, unto its Circumference, A must also be the Center (*a*) of the Circle O S. Therefore the Circles L Q, O S, which cut one another, have the same Center. Which contradicts the 5th Proposition. (a) *By the foregoing.*

PROP. XI. Theorem.

IF two Circles touch each other inwardly, a right Line drawn through their Centers (A and I) passes through the Point of Contact (B.) *Fig.* 13.

If you deny it, let the Centers have, if it may be, that Situation, that a right Line passing through them, shall

fall without the Contact B, cutting the Circles in O and L: Let the Center be A and C; and join A B, C B. Because therefore C B, CO are equal, the common AC being added to each of them, A C+C B shall be equal to A O. But A C, C B are (*b*) greater than A B, that is, than A L (*c*). Therefore also A O is greater than A L, a Part than the Whole. Which is absurd.

(b) *Per* 20. *l.* 1.
(c) *By the Definition of a Circle.*

PROP. XII. Theorem.

Fig. 14.

IF Circles touch one another on the outside, a right Line, which joins the Centers, must pass through the Point of Contact.

If it be denied, let the Centers be so placed, as for instance in A and B, that the Line passing through them shall not pass through the Contact S, but cut the Circles in O and Q. Let the Points A, S and B, S be joined. Then A S, B S together will (*d*) be greater than A B. But A S is (*e*) equal to A O, and B S equal to B Q. Therefore A O and B Q together will be greater than A B, a Part than the Whole. Which cannot be.

(d) *Per* 20. *l.* 1.
(e) *By the Definition of a Circle.*

[Corollary. " A right Line drawn from the Center of " one of the Circles through the Point of Contact, will " pass through the Center of the other.]

PROP. XIII. Theorem.

Fig. 15, 16.

CIrcles touch both one another, and a right Line, in a Point only.

Fig. 15.

For let two Circles touch one another inwardly in a Part of the Circumference L C, if it may be: Then a right Line drawn through the Centers A and B will (*f*) pass through the Point of Contact, as in C. Let there be drawn also A L, B L. Because therefore B L, B C are equal (for they are drawn from the Center B unto the Circumference O L C) the common Line A B being added, A B, B L shall be equal to AC. But AC is equal to A L, for they are both drawn from the Center A unto the Circumference L Q C. Therefore A B, B L are equal to A L, contrary to *Proposition* 20. *L.* 1.

(f) *Per* 11. *l.* 3.

Then

Then let the two Circles touch one another on the outside, in the Arch O L, if it may be. The right Line A P, joining the Centers, will pass through the Point of Contact (*a*) as in O, for instance: Let A L, P L, be drawn. The two Sides of the Triangle A L, P L, will be equal to A O, P O, or the whole A P; contrary to *Proposition* 20. *L.* 1. Fig. 16. (a) *Per* 12. *l.* 3.

Lastly, Let the right Line BF, and the Circle touch each other, if it may be, in some Part (C E:) Let there be drawn unto the Center the right Lines C A, E A. The Lines CA, E A will then be equal: And therefore the Triangle C A E is an *Isosceles*. Wherefore the Angles C and E (*b*) are acute. And therefore a Perpendicular let fall unto B F from the Center A, will fall betwixt E and C, (*c*) as, for instance, in D There will therefore both AC and A E be equal to the Perpendicular A D, which is absurd, and contrary to *Corollary* 14. *p.* 32. and to *Proposition* 47. *L.* 1. (b) *Per Corol.* 11. *Prop.* 32. *l.* 1. (c) *Per Corol.* 3. *Prop.* 32. *l.* 1.

Corollary.

Circles, whose Centers are in the same right Line, and which cut it in the same Point B, do touch one another in that Point only. Fig. 17.

This Proposition is manifest from the very Notion of the Lines which are compared together For neither can a right Line and the curve Circumference of a Circle, or the divers Curvatures of unequal Circumferences, or two Curvatures both convex, agree as to any Part of themselves. But they would agree if they touched one another in some entire and proper Part.

PROP. XIV. Theorem.

In a Circle, equal right Lines (B C, F L) are equally distant from the Center (A.) And what Lines are equi-distant from the Center are equal. Fig. 18.

From the Center (A) let there be drawn (A C, A F.) Likewise A O, A I) at right Angles to B C, F L. Thus B C, F L shall be bisected (*d*) in O and I. (d) *Per* 3. *l.* 3.

Seeing therefore the whole Lines B C, F L are supposed equal, the halves also O C, I F must be equal, and consequently the Squares of them are also equal. Seeing

therefore the Squares of AC, AF are equal, and the Square of AC is equal to O C q, and O A q, as alſo the Square of A F is equal to I F q, and I A Q. (*a*) : It follows, that the two Squares O C q, O A q are equal to the two Squares I F Q, I A Q. Wherefore taking away the Squares of OC, IF (which before were ſhewed to be equal) the Square of A O remains equal to the Square of A I. Therefore the Perpendiculars O A, A I are equal. Therefore (*b*) B C, F L are equi-diſtant from the Center. Which was the firſt Part. Then for the converſe of it;

(a) *Per* 47. *l.* 1.

(b) *Per Def.* 4. *l.* 3.

If the Diſtances A O, A I are ſuppoſed equal, then the Squares of the equal right Lines being taken away, by the ſame Ratiocination it will be ſhewed, that the remaining Squares O C q. I F q are equal, and conſequently that the right Lines O C, I F are equal, which ſeeing they are * halves of the right Lines B C, F L, theſe alſo muſt be equal. Which was the ſecond Part.

* *Per* 3. *l.* 3.

PROP. XV. Theorem.

Fig. 19.

OF right Lines deſcribed in a Circle, the greateſt is the Diameter; and of the reſt, that is the greateſt, which is the neareſt to the Center.

Let there be any Line, as R S different from the Diameter F L. From the Center draw A R, A S. The two, A R, A S, which are equal to the Diameter, are (*c*) greater than R S. Therefore, *&c.*

(c) *Per* 20. *l.* 1.

Then let B I be nearer to the Center than X Z. From the Center unto them draw the Perpendiculars A C, A Q. A Q ſhall be greater (*d*) than A C. Take therefore A O equal to A C, and through O draw R S perpendicular to A O, which (*e*) will be equal to B I; and let A R, A S, A X, A Z be join'd. Becauſe therefore A is the Center, the Sides A R, A S ſhall be equal to A X, A Z. But the Angle RAS is greater than the Angle X A Z. Therefore the Baſe R S, that is, B I, is greater than the Baſe X Z (*f*) Q. E. D.

(d) *Per Def.* 4. *l.* 3.

(e) *By the foregoing.*

(f) *Per* 24. *l.* 1.

PROP. XVI. Theorem.

Fig. 20.

A Right Line (IF) which being drawn through the Point (B,) the Extremity of the Diameter (CB) is perpendicular thereto, falleth all

of it without the Circle, and toucheth it in (B.) Neither can any right Line be drawn betwixt it ſelf and the Circle unto the Point of Contact (B,) but it ſhall cut the Circle.

Part I. Let there be taken in the Line I B F any Point L, unto which, from the Center A, draw the Line A L. Becauſe, in the Triangle A B L, the Angle A B L is a right one, by the Hypotheſis, A L B ſhall be acute (*g*). Therefore A L, which is oppoſite to the greater Angle B, will be greater than A B, which is oppoſite to the leſſer Angle L (*h*.) But A B reacheth only to the Circumference. Therefore A L ſhall reach beyond the Circumference; and conſequently fall without the Circle. Which was the firſt Part.

(g) *Per Corol.* 5. *p.* 32. *l.* 1.
(h) *Per* 19. *l.* 1.

Part II. Below B F, if it may be, let R B fall wholly without the Circle. Becauſe F B A is a right Angle by the Hypotheſis, R B A will be acute, and therefore A B is not perpendicular to B R. Therefore let there be drawn from the Center A to B R, the Perpendicular A O, which (*a*) will fall towards R, and cut the Circle in Q. Therefore A B, which is oppoſite to the greater Angle A O B, is greater than A O, which is oppoſite to the leſſer, to wit, the acute Angle O B A. But A B is equal to A Q: Therefore A Q alſo is greater than A O, a Part than the Whole.

(a) *Per Corol.* 3. *Prop.* 32. *l.* 1.

Corollary.

1. HEnce it appears again, that the Contact of a right Line and a circular one, is only in one Point. *Fig.* 20.

2. If from Centers taken in the ſame right Line infinitely protracted, there be deſcribed through B infinite Circles, as well leſſer than the firſt B S C, as greater; they ſhall all touch the right Line I F in the ſame one Point B. *Fig.* 17.

3. Circles therefore growing into an Amplitude greater than any given one, approach always, even unto Infinity, nearer and nearer to the Tangent, but are never join'd to it, otherwiſe than in one ſingle Point of Contact; which thing, although it be moſt evident, is yet truly admirable.

Fig. 17. 4. From these Things it is manifest, that every Geometrical Line whatsoever is infinitely divisible. For let there be drawn from some Point of the Diameter unto the Tangent the right Line A Q. Infinite Circles having Centers in the right Line B A infinitely produced touch the right Line I F by *Corollary* 2. of this, and one another by *Corollary*, p. 13. in one and the same Point B, and consequently are no where joined, either amongst themselves, or with the right Line I F, but in the Point B only. Therefore it is necessary that they divide the right Line A Q into infinite Parts, that is, into Parts exceeding any Number assignable.

Fig. 20. 5. The Angle of Contingence or Contact L B Q, (that, to wit, which is contained under the Tangent and the Circumference) cannot be divided by any right Line.

Fig. 17. 6. Nevertheless, by Circumferences touching the Line I F in the same Point, it may be divided and diminished infinitely. And in this, and the third *Corollary*, lies hid the whole Mystery of Asymptotes, that is, of a right Line approaching unto an Hyperbola, together with it self infinitely produced, unto a Distance less than any given one, yet never concurring with it.

PROP. XVII. Problem.

Fig. 26. *FROM the given Point (B,) to draw a right Line, which shall touch a given Circle (O Q.)*

From A the Center of the given Circle, let there be drawn into the Point B, the right Line A B, cutting the Periphery in O. From the Center A describe through B another Circle B C, and from O draw O C perpendicular to A B, which may meet the other Circle in C. Draw C A meeting the Circle O Q in I. The right Line drawn from B unto I, will touch the Circle O Q.

For because the Sides B A, I A, are equal to the Sides C A, O A, and the Angle A contain'd betwixt the equal Sides is common to both. In the Triangles I A B, O A C, the Angles A O C, A I B are also (*a*) equal. Therefore A I B is a right Angle. For A O C is a right one by the Construction. Therefore B I (*b*) toucheth the Circle in I.

(a) *Per* 4. *l.* 1.
(b) *Per* 16. *l.* 3.

Scholium.

Scholium.

BY the 31ſt following, from the given Point O, a Line touching a given Circle (B Q) may be well drawn thus : *Fig.* 27.

Let the right Line, joining the given Point O, and the Center A, be biſected in P. Then from the Center P, through A and O, deſcribe a Circle, meeting the given one in B. The right Line O B will touch the Circle.

For A B being join'd, the Angle A B O in the Semi-circle is a right one by *Prop.* 31. Therefore by *Prop.* 16. O B toucheth the Circle B Q.

PROP. XVIII. Theorem.

IF a right Line (CL) touch a Circle, a right Line (AB) drawn from the Center (A) unto the Point of Contact (B) is perpendicular to the Tangent. *Fig.* 28.

If it be denied, let ſome other right Line (as A F) be the Perpendicular from the Center A. This will cut the Circle in O Becauſe therefore the Angle A F B is ſuppoſed to be a right one, A B F (*c*) muſt be acute. Therefore A B (that is, A O) is greater than A F (*d*); a Part than the Whole, which is abſurd.

(c) *Per Corol.* 5. *p.* 32. *l.* 1.
(d) *Per* 19. *l.* 1.

PROP. XIX. Theorem.

IF a Line (B C) touch the Circle, and from the Point of Contact (A,) there be rais'd (A I) perpendicular to the Tangent, the Center will be in that Perpendicular. *Fig.* 29.

If you deny it, let the Center be without A I in Z; and from it let there be drawn unto the Contact the Line ZA. The Angle Z A C will be a right one (*e*) and therefore equal to the Angle I A C, which, by the Hypotheſis, is a right one; that is, the Part will be equal to the Whole, which is abſurd.

(e) *By the foregoing*

PROP. XX. Theorem.

Fig. 30, 31, 32. *THE Angle at the Center (B A C) is double to the Angle (B F C) which is at the Circumference, when the same Arch (B C) is the Base of the Angles.*

Fig. 30. Here are three Cases. In the first Case, the Sides B A, B F coincide. And then because A F, A C drawn from the Center are equal, there will be in the Triangle Z, the Angles F and C equal (*a*). But B A C is equal to the two Angles F and C (*b*). Therefore B A C is double of F.

(a) *Per* 5. *l.* 1

(b) *Per* 32. *l.* 1.

Fig. 31. In the second Case, B A, C A fall within B F, C F, and then F A X being drawn, X A B, by the first Case, is double of X F B; and X A C double of X F C. Therefore the whole B A C is double of the whole B F C.

Fig. 32. In the third Case, B F cuts A C, and the Angle B A C is without the Triangle B F C. Here let F A L be drawn: By the first Case, the whole L A C is double of the whole L F C, and L A B taken away, is double of L F B taken away. Therefore the remaining Angle, B A C, is also double of the remaining one, B F C. *Q. E. D.*

Fig. 53. Corollary. " Hence we gather, that the Sides of every " Triangle are to each other as the Sines of the Angles op- " posite to those Sides respectively. Let E F G be any " Triangle; about which let a Circle be understood to be " circumscrib'd (*c*,) and from the Center of the Circle, " let there be let down the Perpendiculars A B, A C, A D, " which will * bisect the Subtenses. Now as E F is to " E G, so half E F (that is, E B) to half E G (that is, " E D.) But E B is the Sine of the Angle † B A E, that " is, of half the Angle E A F, that is, of the whole Angle " E G F, * opposite to the Side E F; and E D is the Sine " of the Angle E A D, that is, of half the Angle E A G, " that is, of the whole Angle E F G, which is opposite " to the Side E G. Therefore E F is to E G, as the Sine " of the Angle E G F, is to the Sine of the Angle E F G. " *Q. E. D.* And from this one Proposition a great Part of " Trigonometry is deduced. Which thing will be worth " our Observation.

(c) *Per* 5 *l.* 4.

* *Per* 3. *l.* 3.

† *Per Corol.* 2. *p.* 3. *l.* 3.

* *Per* 20. *l.* 3.

Corollary (2.) " From the former Corollary we learn *Fig.* 86.
" to meaſure the Diſtance of the Moon. For Aſtronomi- *l.* 1.
" cal Obſervations giving us the Angle of the Diurnal Pa-
" rallax * B C A, we find out the Diſtance of the Moon by * *Corol.* 16.
" the following Proportion. As the Sine of the Angle *p.* 32. *l.* 1.
" A C B, is to the Sine of the Angle A B C; ſo is the Semi-
" diameter of the Earth, B A, unto the Moon's Diſtance,
" A C. *Q. E. I.*

Corollary (3.) From the ſecond Corollary we learn alſo *Fig.* 54.
" to meaſure the Diſtance of the Sun, for there being given
" by Aſtronomical Obſervations the Angle of the Men-
" ſtrual Parallax, (namely, that which is made when the
" Moon appears preciſely biſected) or the Angle Z E O,
" and, together with this Angle, the Moon's Diſtance, Z O.
" We find the Diſtance of the Sun by this Analogy. As
" the Sine of the Angle Z E O, is to the Sine of the Angle
" E O Z; which Sine is the Radius: So is Z O, the Moon's
" Diſtance, unto Z E, the Diſtance of the Sun. *Q. E. I.*

PROP. XI. Theorem.

THE Angles (B Q C, B F C) which in a Circle ſtand upon the ſame Arch (B O C,) or which are in the ſame Segment (B Q S C) are all equal among themſelves. *Fig.* 33.

Let firſt the Segment B Q S C be greater than a Semi-circle. From the Center A draw A B, A C. By the fore-going, the Angle B A C, at the Center, is double of each, B Q C, B F C. Therefore they all, B Q C, B F C, are equal (*a.*) *Q. E. D.* (a) *Per Axiom* 6.

Then let the Segment B Q C be equal to, or, leſs than a Semi-circle. In the Triangles B Q I, C F I, becauſe the Angles vertically oppoſite at I are equal (*b,*) the Sum of the reſt, Q and R will be equal to the Sum of the (*c*) reſt, F and O. Wherefore, if from theſe equal Sums there be taken away the Angles R and O, which by the firſt Part, are equal, as ſtanding upon the ſame Arch Q F, the Angles which remain, Q F, muſt be equal. *Q. E. D.* *Fig.* 34. (b) *Per* 15. *l.* 1. (c) *Per Corol.* 10. *p.* 32. *l.* 1.

Corollary. " Hence we gather in Opticks, that any
" Line B C to the Eye placed where you will in the Cir-
" cumference

" cumference of the Circle, whereof the Lines is a Chord,
" appears of the fame Magnitude; to wit, becaufe it ap-
" pears every where under an equal Angle B Q C.

Scholium. " If of two equal Angles, ftanding upon " the fame Arch, one of them be at the Circumference, " the other alfo will be at the Circumference.

Fig. 33, 34. " If it be denied, B Q C fhall either be equal to the " Angle B I C, on this fide the Circumference Q F, or to " the Angle BEC, which is beyond the faid Circumference. " But the Angle BIC, is (*d*) greater, and the Angle BEC " (*d*) is lefs than the Angle B Q C. Therefore, &c.

(d) *Per Corol.* 1. *p.* 32. *l.* 1.

PROP. XXII. Theorem.

Fig. 35. *IN any Quadrilateral infcribed in a Circle (A B C F) the oppofite Angles make two right ones.*

Let B F, C A be drawn. The Angle A B C, with the (*a*) two, O and X, make two right Angles. But O is equal to I (*b*,) becaufe it ftands upon the fame Arch, B C: And again, X is equal to Z, becaufe it ftands upon the fame Arch, A B. Therefore A B C taken together, with the two Angles I and Z, that is, with the whole oppofite Angle A F C, makes two right Angles. *Q. E. D.*

(a) *Per* 32. *l.* 1.
(b) *Per* 21. *l.* 3.

Corollary (1.) " Hence, if one Side of a Quadrilateral, " defcribed in a Circle, be protracted, the external Angle " will be equal to the oppofite Angle of the Quadrilateral; " for the internal, added to either of them, makes two " right Angles.

(2.) " Likewife a Circle cannot be defcribed about a " Rhombus, becaufe its oppofite Angles either fall fhort of, " or exceed two right Angles.

(3.) " Likewife, if in any Quadrilateral A B C F, the " oppofite Angles F and B are equal to two right ones, a " Circle may be defcribed about it. For (*a*) a Circle will " pafs through any three Angles, C, F, A, and this, fo that " * fourth be equal to B; which cannot be, unlefs it doth " indeed pafs through the Point B †, Therefore it doth " pafs through it.

(a) *Per* 5. *l.* 4.
* *Per* 22. *l.* 3.
† *Per Schol. p.* 21. *l.* 3.

PROP.

PROP. XXIII, XXIV. Theorems.

ARE not neceſſary; and they treat of ſimilar Segments, which cannot rightly be defin'd without Proportions.

PROP. XXV. Problem.

TO perfect a given Arch (ABC.) *Fig.* 36.

Let there be ſubtended at Random the two right Lines A B, C B; which biſect in I and L. From I and L raiſe Perpendiculars meeting one another in O. This ſhall be the Center of that Circle whereof A B C is a Portion.

For (*a*) the Center is both in the Line I X, and in the Line L Z. Therefore it is in their common Point O. (a) *Per Corol. p.* 1. *l.* 3.

The Practice. From the Center B, taken in the Arch, deſcribe a Circle; and with the ſame Interval, from other Centers in the Arch, deſcribe two other Circles, each of which cuts the former twice. Two right Lines drawn through the Interſections, and croſſing each other in O, will give the Center.

PROP. XXVI, XXVII. Theorems.

IN equal Circles, equal right Lines (CE, FI) ſubtend equal Arches; and if the Arches are equal, the Subtenſes are alſo equal. *Fig.* 37.

Theſe two Propoſitions are plainly Axioms, and need no Demonſtration.

Corollary (1.) " If in a Circle A B C D the Arch AB be *Fig.* 55.
" equal to the Arch D C; A D will be parallel to B C.
" For A C being drawn, the Angles A C B, C A D, as
" ſtanding on equal Arches, will be equal. Wherefore * *Per* 27.
" A D is parallel to B C. *Q. E. D.* *l.* 1.
(2.) " The right Line E F, which is drawn from the *Fig.* 56.
" Point A, the middle Point of ſome Arch, and toucheth
" the

" the Circle, is parallel to the right Line B C, which subtends that Arch. For from the Center D, draw unto the Point of Contact A the right Line DA, and join DB, DC. The Side D G is common, and D B is equal to D C, and the Angle B D A equal to the Angle C D A, the Arches BA, CA being supposed to be equal. Therefore the Angles D G B, D G C are equal *, and consequently are right Angles But the internal Angles GAE, GAF are also right Angles †. Therefore B C, E F are parallel *. *Q. E. D.*

* Per 4. l. 1.
† Per 18. l. 3.
* Per 28. l. 1.

PROP. XXVIII, XXIX. Theorems.

Fig. 38.

If in equal Circles, the Angles, whether at the Centers (B A C, F L I) or at the Circumference (B O C, F S I) be equal; the Arches also (B X C, F Z I) on which they stand are equal; and if the Arches be equal, the Angles also are equal.

These two Propositions also are plainly Axioms, and need no Demonstration.

PROP. XXX. Problem.

Fig. 39.

To bisect a given Arch (A B C.)

Draw A C, which bisect in O. From O draw the Perpendicular O B, meeting the Arch in B. I say the thing is done.

For let A B, B C be join'd. The Sides A O, B O are by the Construction equal to C O, O B; and the Angles at O are equal, as being right ones. Therefore the Bases A B, C B are equal (*a*). Therefore the Arches also (*b*) A B, B C are equal.

(a) Per 4. l. 1.
(b) Per 26. l. 3.

The Practice. From the Centers A and C describe with an equal Interval, Arches cutting each other in the Point F and I, the right Line drawn through these Points will bisect the Arch A B C.

PROP.

PROP. XXXI. Theorem.

THE Angle (BCF) in a Semi-circle, is a right one; that in a Segment greater than a Semi-circle, is less than a right one; that in a Segment less than a Semi-circle, is greater than a right one. Fig. 40.

Part I. From the Center A draw AC. Because AB and AC are equal, the Angles O and B are equal (*c*.) (c) *Per* 5.
For the same Cause the Angles I and F are equal. *l.* 1.
Therefore the Angle BCF is equal to B and F together. Seeing (*a*) therefore the three together make two right (a) *Per* 32. Angles, BCF, which is half of two right Angles, is one *l.* 1. right Angle.

Part II. Let the Segment LOBC be greater than a Fig. 41. Semi-circle, and in it let there be the Angle COL, and let LB, the Diameter of the Circle, be drawn. The Angle COL is less than that BOL, which, by Part I. is a right one. Therefore, &c.

Part III. Let the Segment LOX be less than the Semi- Fig. 41. circle LOB, and XOL be the Angle in it. This will be greater than BOL, which is a right one. Therefore, &c.

Corollary (1.) "Hence we may make a Proof of the Fig. 40.
"Instrument, called a Square, whether it be exactly Rect-
"angular or not. For in what Circle soever the top of the
"Square is laid upon C, or any Point of the Circumfe-
"rence whatsoever, if the Sides of it do pass through the
"Points of the Diameter B and F, the Angle is a right
"one; otherwise not.

(2.) "If the Sides of a Square be held continually upon
"the Points B and F, in the mean while that the Angle is
"moved round, first on one Side, then on the other, the
"top of the Angle C will describe a Circumference of a
"Circle, whose Diameter is the Line BF.

(3.) "Hence we learn to raise a Perpendicular at the
"end of a Line. Let BC be the Line, C the Point given,
"from whence a Perpendicular is to be rais'd. From any
"Point whatsoever, A, as the Center, let a Circle be described
"passing through the Point C, and cutting BC in any Point,

" as B. If the Diameter B F be drawn, it is manifest that " the Line CF is the Perpendicular required. *Q. E F.*

Fig. 57. (4.) " Hence it is manifest, that Circles touching one " another inwardly, do cut all Lines, as A D proportionably; or so, that A C, the Subtense of the lesser, is to " A D, the Subtense of the greater Circle; as A E, the " Diameter of the lesser, is to A B, the Diameter of the " greater. For there being drawn the Subtenses E C, B D, " the Triangles E A C, D A B are equi-angled. For the " Angle A is common, and A C E, A D B are right ones, " as being Angles in a Semi-circle; and therefore A E C, " A B D (*b*) are equal. The Triangles therefore are similar, by the Fourth Proposition of the Sixth Book, and " A E : A B : : A C : A D. *Q. E. D.*

(b) *Per Corol.* 9. *Prop.* 32. *l.* 1.

Fig. 40. (5.) " In a right-angled Triangle B C F, if the Hypothenuse B F be bisected in A, the right Line A C cuts " the Triangle into two equicrural ones, A C B, A C F, " and so a Circle described from the Center A, through " B, must pass through C, the Top of the right Angle.

PROP. XXXII. Theorem.

Fig. 42, 43. ***If a right Line (CF) touch a Circle, and another (AB) which is drawn from the Point of Contact (A) cut it, the Angle made by the Tangent and the cutting Line, is equal to the Angle which is made in the internal or opposite Segment.***

That is, the Angle C A B will be equal to the Angle L, which is made in the Segment A L B; and the Angle F A B will be equal to the Angle O, which is made in the Segment A O B. For,

Fig. 42. First, let the Line A B pass through the Center. Here by *Prop.* 18. C A B is a right Angle: And by *Prop.* 31. L is also a right one. Therefore C A B and L are equal.

Fig. 43 Then let the Line A B not pass through the Center. Let the Line A Q therefore be drawn through the Center, and B Q be join'd. Because the Angle in the Semi-circle A B Q (*a*) is a right one, B Q A taken together with B A Q will make one right Angle (*b.*) But C A Q is also by *Prop.* 18. of this Book a right Angle. Therefore B Q A with B A Q, are equal to C A Q. The common Angle therefore B A Q being

(a) *Per* 31. *l.* 3.
(b) *Per* 32. *l.* 1.

being taken away, there remains B Q A, which is equal to L (*c*) equal to CAB. Therefore L and C A B are equal: Which is the firſt Part to be proved. (c) *Per* 21. *l.* 3.

Then F A B and C A B make two right Angles (*d*,) and in the Quadrilateral B O A L, the Oppoſites L and O make two right Angles (*e*.) Therefore the two F A B, C A B are equal to the two O and L. Therefore there being taken away on one Side C A B, on the other L, which have already been ſhew'd to be equal, there remains F A B equal to O. Which was the other Part to be proved. (d) *Per* 13. *l.* 1. (e) *Per* 22. *l.* 3.

PROP. XXXIII. Problem.

UPON a given Line (B C) to make a Segment of a Circle, in which the Angle ſhall be equal to any Angle given. Fig. 44.

Firſt, let there be an acute Angle given, A B F, from B draw B L perpendicular to A B; and at C, the Extremity of the Line B C, make B C I equal to C B L (by 23. *l.* 1.) whoſe Side ſhall cut B L in I. From the Center I deſcribe a Circle through B: This Circle will alſo paſs through C (foraſmuch as becauſe of the Equality of the Angles at B and C, the Sides likewiſe C I, B I are (by 6. *l.* 1) equal, and the Segment B Q C ſhall contain an Angle equal to the given one A B F.

For becauſe A B is perpendicular to the Diameter B L, A B will touch the Circle which B C cuts (*a*.) Therefore the Angle in the Segment B Q C is equal (*b*) to the Angle A B F. (a) *Per* 18. *l.* 3. (b) *By the foregoing.*

But if the Angle given be obtuſe, as R B C, do as before, and C O B will be the Segment required.

PROP. XXXIV. Problem.

FROM a given Circle to take away a Segment, containing an Angle equal to a given one. Fig. 45.

Unto the Diameter of the Circle F A, draw the Perpendicular B A L. Then (*e*) let A C be drawn, which may make the Angle B A C equal to that which is given. This Line A C ſhall cut off the Segment A Q C, containing an Angle equal to the given one, as is manifeſt from *Prop.* 32. (e) *Per* 23. *l.* 1.

PROP.

PROP. XXXV. Theorem

Fig. 46, 47, 48. *If in a Circle two right Lines (CL, BF) cut one another, the Rectangle COL) under the Segments of one is equal to the Rectangle (BOF,) under the Segments of the other.* For,

If they interfect each other in A, the Center of the Circle, the thing is manifeft.

Fig. 46. If one of them CL paffeth through the Center A, and bifects the other BF, which doth not pafs through the

(f) *Per* 3. *l.* 3. Center; it (*f*) cuts it perpendicularly, and fo the Square of FO is the fame with the Rectangle FOB. Let AF be drawn. Becaufe CL is bifected in A, and otherwife divided in O.

It will be thus,

(a) *Per* 5. *l.* 2.

$$\left.\begin{array}{l}\text{Rect. } COL \\ + AOq.\end{array}\right\} \text{will be equal to } ALq\ (a.)$$

that is, AFq.

(b) *Per* 47. *l.* 1.

$$\text{that is, equal to } \left.\begin{array}{l}AOq. \\ + FOq.\end{array}\right\}\ (b.)$$

Therefore the common Square AO being taken away, there will remain

Rect. COL equal to FOq. that is,
to the Rect. FOB.

Fig. 47. Then if one of the right Lines CL paffes through the Center, and cuts the other BF unequally in O, let a right Line drawn from the Center A cut BF into two equal Parts

(c) *Per* 3. *l.* 3. in I. In this Cafe, AIB will be a right Angle (*c.*) Now, becaufe CL is bifected in A, and otherwife in O, it will be thus,

(d) *Per* 5. *l.* 2.

(e) *Per* 47. *l.* 1.

$$\left.\begin{array}{l}\text{Rect. } COL \\ + AOq.\end{array}\right\} \text{will be equal to } ALq.\ (d) \text{ that is, to } ABq \text{ that is, to } \left.\begin{array}{l}AIq. \\ + BIq.\end{array}\right\}\ (e.)$$

(f) *By the fame.* But AOq is equal to OIq + AIq. (*f.*) Therefore.

$$\left.\begin{array}{l}\text{Rect. } COL \\ + OIq. \\ + AIq.\end{array}\right\} \text{equal to } \left.\begin{array}{l}AIq. \\ + BIq.\end{array}\right\}$$

Therefore the common Square AI being taken away there remains,

Rect

$\left.\begin{array}{l}\text{Rect. COL}\\ +\text{OI q.}\end{array}\right\} = \text{BI q.}$

But B I Square is equal to the Rectangle F O B, together with O I Square; (*g*) because F B is bisected in I, and otherwise cut in O. Therefore, (g) *Per* 5. *l.* 2.

$\left.\begin{array}{l}\text{Rect. COL}\\ +\text{OI q.}\end{array}\right\}$ are equal to $\left.\begin{array}{l}\text{Rect. FOB}\\ +\text{OI q.}\end{array}\right\}$

Therefore the common O I q being taken away, there remains,

Rect. C O L. = Rect. F O B

But lastly, if neither of the Lines C L, F B passes through the Center: Through their common Intersection (O) let there be drawn the right Line X Z, which passes thro' the Center (A.) By what hath been just now demonstrated, both the Rectangle C O L, and that F O B, are equal to the Rectangle Z O X. Therefore C O L, F O B, are equal betwixt themselves (*h*.) *Fig.* 48. (h) *Per Axiom* 1.

[" Or the Proposition may be demonstrated more easily " and universally thus. Join A C and B D. Here, be- " cause of the Equality of the Angles C E A,, B E D, as " being vertically opposite (*a*); and of the Angles C and " B, as being upon the same Arch A D †: the Triangles " C E A, B E D are equi-angled (*per Corol.* 9. *p.* 32. *l.* 1.) " Therefore * C E : E A : : E B : E D. Therefore C E " ×E D is equal to E A×E B (*per* 16. *l.* 6.) *Q. E. D.* *Fig.* 58. (a) *Per* 15. *l.* 1. † *Per* 21. *l.* 3. * *Per* 4. *l.* 6.

PROP. XXXVI. Theorem.

IF from (B) a Point given without a Circle, there be drawn unto the Circle two right Lines, one (BF) touching it, the other (BC) cutting it; the Rectangle (CBO,) which is comprehended under the whole cutting Line (C B) and the Part (B O,) which lies betwixt the Point (B) and the Circle, is equal to the Square of the Tangent (BF.) *Fig.* 49, 50, 51.

1. If the cutting Line B C passes through the Center A, join A F. This, with the Line F B, will make a right Angle (*a*.) And therefore, because C O is bisected in A, and to it is added O B; it will be thus, (a) *Per* 18. *l.* 3.

$\left.\begin{array}{l}\text{Rect. CBO}\\ +\text{AO q}\end{array}\right\}$ will be equal to A B q (*b*) that is, (b) *Per* 6. *l.* 2.

to $\left.\begin{array}{l}\text{AF q}\\ +\text{FB q}\end{array}\right\}$ (*c*.) (c) *Per* 47. *l.* 1.

 There-

Therefore the equal Squares AOq, AFq being taken away on both Sides, there remains,

Rect. CBO,=BFq.

Fig. 50, 51. 2. But then, if CB doth not pass through the Center, let there be drawn AB, AF, AO, and AL, and let AL bisect OC in L. The Angle ALO is therefore a right one (*d.*) Likewise AFB is a right Angle (*e.*) Now, because CO is bisected in L, and to it is added OB, it will be thus,

$$\left.\begin{matrix}\text{Rect. CBO}\\ +\text{LOq}\end{matrix}\right\} = \text{LBq } (f.)$$

Let there be added on both Sides AL Square, and then

$$\left.\begin{matrix}\text{Rect. CBO}\\ +\text{LOq.}\\ +\text{ALq.}\end{matrix}\right\} \text{equal to } \left\{\begin{matrix}\text{LBq.}\\ +\text{ALq.}\end{matrix}\right.$$

But the Squares of LO, AL are equal (*g*) to the Square of AO, or AF; and the Squares of LB, AL are equal to the Square of AB (*h.*) Therefore,

$$\left.\begin{matrix}\text{Rect. CBO}\\ +\text{AFq.}\end{matrix}\right\} = \text{ABq. that is, } (i) \text{ to } \left\{\begin{matrix}\text{BFq.}\\ +\text{AFq.}\end{matrix}\right.$$

(d) *Per* 3. *l.* 3.
(e) *Per* 18. *l.* 3.
(f) *Per* 6. *l.* 2.
(g) *Per* 47. *l.* 1.
(h) *By the same.*
(i) *By the same.*

Therefore the common Square, that of AF being taken away, there remains

Rect. CBO equal to the Square of BF. *Q. E. D.*

Fig. 59. [" Or more easily and universally thus. Draw AB and
" BC. Now because of the Equality of the Angles A.
" and DBC, * and for that the Angle D is common to
" both; the Triangles BDC, ADB are equi-angled †.
" And therefore (by *Lib.* 4. 6.) AD : DB : : BD : DC.
" Wherefore the Rectangle * AD × DC is equal to the
" Rectangle BD × DB, or DBq. *Q. E. D.*]

* *Per* 32. *l.* 3
† *Per Corol.* 9. *p.* 32. *l.* 1.
* *Per* 17. *l.* 6.

Corollaries.

Fig. 52. 1. IF from the same Point B, without the Circle, as many cutting Lines BC as you will be drawn, all the Rectangles CBO are equal amongst themselves. For each of them is equal to the Square of the Tangent BF.

2. The right Lines BF, BQ, which, from the same Point, touch the Circle, are equal. For each of their Squares is equal to the same Rectangle.

3. " It

3. " It is alſo clear, that from the ſame Point B, taken
" without the Circle, there can only two Lines B F, B Q
" be drawn, which ſhall touch the Circle. For if a third
" be ſaid to touch it, it muſt be equal to B F, or B Q,
" and therefore the ſame with one of them.

4. " In every Right-angled Triangle B F A, that is *Fig.* 44.
" not alſo an Iſoſceles, the Rectangle ariſing from the Sum
" of the Hypothenuſe, and one Side drawn into the Dif-
" ference betwixt them, is equal to the Square of the other
" Side. For the Sum of the Hypothenuſe B A, + A F or
" A C, is = B C. And their Difference is B A — A F = B A
" — A O = B O. And the other Side of the Triangle is
" B F. But the Rectangle C B O is equal to the Square
" of B F. Therefore, &c.

PROP. XXXVII. Theorem.

If the Rectangle under C B and O B be equal to the Square of B F, this muſt touch the Circle in F. Fig. 52.

From B let there be drawn the Tangent B Q, and the right Lines E Q, E F being drawn from the Center E, unto the Points Q and F, let B E be joined. Becauſe by the Suppoſition the Square of B F is equal to the Rectangle C B O, as is alſo the Square of B Q, by 36th of this Book, the Squares of B Q, B F ſhall be equal betwixt themſelves, " and conſequently the right Lines B Q, B F are equal. Therefore the Triangles F E B, B E Q are Equilateral each to other. Therefore the Angles Q, F are equal (*a.*) But Q is a right Angle (*per* 18. *l.* 3.) Therefore F alſo is a right Angle. Therefore B F toucheth the Circle (*b.*)

(a) *Per* 8. *l.* 1.
(b) *Per* 16. *l.* 3.

Corollaries (1.) " Hence the Angle E B F is equal to
" the Angle E B Q (*per* 8. *l.* 1)

(2.) " If two equal right Lines B F, B Q fall from
" ſome Point B upon the convex Circumference, and B F,
" one of them toucheth the Circle, the other B Q muſt
" touch it alſo. For ſeeing B F, B Q are equal, their
" Squares are alſo equal. But B F q is equal to C B O
" (*a*) Therefore B Q q = C B O (*b.*) Therefore B Q
" alſo toucheth the Circle (*c.*)

(a) *By the foregoing.*
(b) *Per Axiom* 1.
(c) *By this Propoſition.*

 Scholium,

Scholium [1.] " Seeing all Planes paſſing through the " Center of the Earth, which all ſtand perpendicular upon " the Horizon, do produce great and equal Circles upon " the Earth's Surface, we ſhall here bring in ſome elegant " Conſectaries from thence, out of our Author in his Aſtro- " nomy; which from the Nature of Circles may very " eaſily be underſtood.

(1.) " If in any Part the Surface of the Earth were " perfectly plane, Men could no more ſtand upright upon " it, than upon the ſide of an Hill, ſaving in the Point of " a Contact only.

(2.) " The Head of a Traveller performs a longer Way " or Courſe than his Feet: Likewiſe he that is on Horſe- " back, and goes the ſame Way as a Footman, meaſures a " greater or longer Space than he that is on Foot. As " likewiſe in a Ship, the uppermoſt Part of the Maſt runs " over more Way than the lower Parts of it.

(3.) " If any one ſhould travel over the whole Circum- " ference of the Earth, the Way gone over by his Head " would exceed that which was gone over, by his Feet, by " the Difference of Circumferences; or by the Circum- " ference of a Circle, whoſe Semi-diameter is the man's " own Stature.

(4.) " If a Veſſel full of Water be elevated perpendicu- " larly, the Water will continually be running over, and " yet it will remain full; namely, becauſe the Surface of " the Water is continually compreſſed into the Surface of a " greater Sphere. Yea, if a Veſſel be elevated continually " higher and higher, the Surface of the Water which is " contained in it, will continually deſcend and come nearer " unto a Plane; unto which yet it will never actually come

(5.) " If a Veſſel full of Water be carried directly " downwards, although nothing run over, yet it will ceaſe " to be full; namely, becauſe the Surface of the Water " ſwells continually into a Part of a leſſer Sphere. From " whence it follows.

(6.) " That one and the ſame Veſſel contains more " Water at the Foot of a Mountain than at the Top; as " likewiſe more in a ſubterraneous Cellar, than in a Cham- " ber. To which Things add.

(7.) " That two Strings, on which two Iron Balls hang " perpendicular, [and conſequently the Walls of Buildings " erected perpendicularly] are not Parallel one to another,

" but

" but Parts of Radius's meeting together in the Center of
" the Earth.

Scholium [2.] " I think it not amiss to insert in this *Fig.* 60.
" Place the following Problem also, which was communi-
" cated to me by a Friend, as demonstrated by me some-
" what more briefly.

" Through the two Points (B) and (C) in a given Circle
" (F D M) to draw the Circumference of a Circle which
" shall bisect the Circumference of the other given Circle.

" Through the Center A, and one of the given Points
" B, let there be drawn the infinite right Line B A M E.
" Unto which, from the Center, let there be erected the
" Perpendicular A D, and let B D be drawn. Let the
" Line D E, be made perpendicular to B D, cutting the
" infinite Line B A M E in the Point E. Lastly, let a (a) *Per* 5.
" Circle be drawn (*a*) through the three Points, B, C, E. *l.* 4.
" I say the Thing is done. For,

" Let the Chord of the second Circle be drawn through
" either of the Intersections of the Circles, as G, and
" through A the Center of the first Circle, to wit, G A f;
" let also the Diameter of the first Circle G A F be drawn.
" Then in the first Circle (by *Corol.* 1. *Prop.* 8. *l.* 6. and
" by *Prop.* 17. *l.* 6.) B A × A E = A D q, that is, (because
" of the Equality of the Semi-diameters, A D, AG, A F)
" A, G × A F. And in the second Circle there will be
" (*b*) A B × A E = A G × A f. Therefore A F = A f, and (b) *Per* 35.
" the Points F and f will coincide, and the Arch F D G is *l.* 3.
" equal to the Arch F M G. *Q. E. F.*

THE

THE

Elements of EUCLID.

BOOK IV.

THIS Book, which is wholly Problematical, teacheth by what Artifice, Figures, thoſe which are ordinate or regular eſpecially, may be inſcribed in, and circumſcribed about Circles. There is very great Uſe of it in building Fortifications; and from it, as a Fountain, have been derived thoſe moſt excellent Tables of Sines, Tangents and Secants, to the very great Benefit of the Mathematicks.

[" This Book is moſt uſeful for *Trigonometry*; for by in-
" ſcribing *Polygons* in a Circle, we learn to frame Tables of
" *Chords*, *Tangents* and *Secants*; by the help of which we
" learn to meaſure the *Magnitudes* of *Figures* and *Bodies*.
" Neither without it can we duly *diſtinguiſh* the *Aſpects*, as
" they call them, of the *Stars*, as the *Quartile*, *Sextile*, &c.
" they wholly depending upon the *Inſcriptions* of *Polygons* in
" *Circles*. Neither can we otherwiſe collect the *Area* (which
" is a certain *Quadrature* of a *Circle*) than from the *Area's* or
" *Squares* of innumerable *Polygons* inſcrib'd in, and circum-
" ſcrib'd about a *Circle*. And in like manner we know
" *duplicate Proportion* of *Circles* amongſt themſelves, from
" the duplicate *Proportion* of *Polygons* inſcrib'd in, or circum-
" ſcrib'd about, *Circles*. And as for *Military Architecture*, it
" makes ſo much uſe of *Polygons* inſcrib'd in *Circles*, that
" more than all other Sciences it may ſeem to be wholly
" owing to this Book.]

DEFINITIONS.

1. A Rectilinear Figure is said to be inscrib'd in a Circle, or to have a Circle circumscrib'd about it, when the Tops of all the Angles thereof are in the Circumference of the Circle.

2. A Rectilinear Figure is said to be circumscrib'd about a Circle, or to have a Circle inscrib'd in it, when each one of its Sides toucheth the Circle.

3. An ordinate or regular Figure is that which is Equilateral and Equiangular

PROPOSITION I. Problem.

To inscribe a right Line (A,) which is not greater than the Diameter, in a Circle (BD.) Fig. 1. l. 4.

Take in the Circumference any Point B. From the Center B, with the Interval of the given Line A, describe an Arch, cutting the Circle in C: Draw the right Line BC. I say the Thing is done.

PROP. II. Problem.

To inscribe in a Circle a Triangle having equal Angles with a given one (X). Fig. 2.

Let the Line EF touch the Circle in D. Let EDG be made (*a*) equal to the Angle C, and FDH equal to B; and join GH. I say the Thing is done. For (*b*) EDG is equal to H. H consequently is equal to the Angle C (*c*.) And FDH is equal (*d*) to G; and consequently G to B. Therefore GDH (*e*) is equal to the Angle A. Therefore what was required is done.

(a) *Per* 23. *l.* 1.
(b) *Per* 32. *l.* 3.
(c) *By the Construction.*
(d) *Per* 32. *l.* 3.
(e) *Per Corol.* 9. *p.* 32. *l.* 1.

PROP. III. Problem.

To circumscribe about a Circle a Triangle, having equal Angles with a given one (IKL.) Fig. 3.

Let the Line IK be drawn forth on both Sides, so as to make the external Angles O and N. Make at the Center A, the Angles GAB, BAF equal to O, N respectively,

which is done by 23. *l.* 1. Then in the Points G, F, B, let three right Lines touch the Circle, meeting together in C, E, D. The Triangle C E D is circumscrib'd about the Circle, and is equi-angled to the given one I L K. For,

In the Quadrilateral C G A B, the Angles G and B are (*f*) both of them right ones. Therefore the remaining ones G A B, and C taken together, do (*g*) make two right ones, and consequently are equal to the two together, O, I. Therefore the two, G A B and O, which are equal by the Construction, being taken away, there remains C equal to I. In the same manner E will be proved equal to the Angle K. Therefore D and L are (*h*) likewise equal. That therefore is done which was demanded.

(f) *Per* 18. *l.* 3.
(g) *Per Theorem* 1. *Schol p.* 32. *l.* 1.
(h) *Per Corol.* 9. *Prop.* 32. *l.* 1.

For that the Tangents do concur is thus shew'd. The Angles O, I, and K, N are (*a*) equal to four right ones; and I K are less than two right ones. Therefore O, N (that is, by the Construction G A B and B A F) are greater than two right ones. Therefore G A F (*c*) is less than two right ones. Therefore G F falls between A and D. Therefore seeing A G D, and A F D are right Angles, D G F, and D F G are less than two right ones. Therefore C G D and E F D † meet together towards D. In the same manner it may be demostrated that the rest concur.

(a) *Per* 13. *l.* 1.
(b) *Per* 32. *l.* 1.
(c) *Per Corol.* 3. *Prop.* 13. *l.* 1.
† *Per Schol. p.* 3. *l.* 1.

Fig. 3.

PROP. IV. Problem.

TO inscribe a Circle in a Triangle.

Fig. 3. Bisect the two Angles C and E with the Lines CA, E A, meeting together in A. From A draw the Perpendiculars, A B, A G, A F. A Circle described from the Center A through B, will pass also through G and F, and touch the three Sides of the Triangle.

For in the Triangles C A G, C A B because the Angles A G C, A B C, and likewise those G C A, and B C A are equal by the Construction, and the Side A C is common, the Sides A G, A B * must be likewise equal. In like manner A B, A F may be shewn to be equal. Therefore the Circle describ'd from the Center A, passeth through B, G, F. And because the Angles at those three Points are equal, it toucheth † all the Sides of the Triangle. That therefore is done which was required.

* *Per* 26. *l.* 1.
† *Per* 16. *l.* 3.

[" Hence

[" Hence the Sides of a Triangle being known, the Seg-
" ments of them, which are made from the Contacts of an
" inscribed Circle, will be known. Let D C be 12. D E
" 18. C E 16. D C and C E will be 28. from which sub-
" tract 18 = D E = D G + B E, there remains 10 = C G
" + C B. Therefore C G or C B = 5. Consequently E B
" or E F = 11. Wherefore F D or D G = 7.]

PROP. V. Problem.

TO describe a Circle about a Triangle, or through three given Points, B, C, D, not lying in a right Line, to describe a Circle. Fig. 4.

Connect the given Points with two right Lines B C, C D, which bisect with the Perpendiculars E A, OA, meeting together in A. This will be the Center of a Circle which passeth through B, C, D.

Let the right Lines A C, A D, A B be drawn. By the Construction, the Sides D O, O A are equal to these, C O, AO; and the Angles at O are right ones. Therefore A D is equal to A C (*a*.) In the same manner A B may be prov'd equal to A C. Therefore A D, A B (*b*) are equal. Therefore a Circle described from the Center A through B, will pass also through C and D. Which was the Thing required.

(a) *Per* 4. *l.* 1.
(b) *Per Axiom* 1.

As for the Practice, it is sufficient to describe from B, C, D, three equal Circles, intersecting each other; and through the Intersections to draw right Lines, these meeting one another will give the Center sought.

PROP. VI, VII. Problems.

TO inscribe a Square in, and circumscribe one about a Circle. Fig. 5.

Let the Diameters B D, C E be drawn, cutting each other perpendicularly. The right Lines which join the Terms of these, inscribe a Square in a Circle.

The

The Demonstration is manifest from 4. l. 1. and 31. l. 3. Then let four Tangents be drawn touching the Circle in B, C, D, E, meeting together in I, F, G, H. The Figure I F G H is a Square, circumscrib'd about a Circle.

The Demonstration is manifest from 18. l. 3. with *Corrollary* 2. *Proposition* 36. l. 3. and 28, and 34. l. 1.

Scholium.

Fig. 5. A Square describ'd about a Circle is double to that inscrib'd. For because the Angle B C D in the Semicircle (c) is a right one, the Square of B D (that is, F I Square shall be (d) equal to B C q. + C D q, and therefore double to the Square of C D, *i. e.* to C D E B.

(c) *Per* 31. *l.* 3.
(d) *Per* 47. *l.* 1.

PROP. VIII, IX. Problems.

Fig. 6. *TO inscribe a Circle in, and circumscribe one about a Square, as (B C F E.)*

Let there be drawn the Diameters of the Square, cutting each other in O. From the Center O describe a Circle through B; this will also pass through E, F, C.

Then from the Center O draw O D perpendicular to B C; a Circle describ'd from the Center O through D, will touch all the Sides of the Square.

Part I. Because, by the Hypothesis, the Lines C B, E B are equal; the Angles B C E, B E C will be equal (c) But C B E is a right Angle by the Hypothesis. B C E therefore and B E C are half right ones (d.) In the same manner C B F will be shew'd to be an half right Angle, as likewise the rest of the Angles; and so they are equal amongst themselves. Therefore in the Triangle B A C, seeing there are two equal Angles C B O, B C O, the right Lines O B and O C (e) are equal. In the like manner the right Lines O B, O E, O F may be shew'd to be equal Therefore a Circle described from the Center O through B, will pass through E, F, C.

(c) *Per* 5. *l.* 1.
(d) *Per Corol.* 11. *p.* 32. *l.* 1.
(e) *Per* 6, *l.* 1.

Part II. From O let there be also drawn the Perpendiculars OG, OH, OI. Because in the Triangles GBO, DBO the Angles at D and G, as likewise those at B, are equal, and th

the Side OB is common, the Sides O D, O G must be equal (*a*) In the same manner O G, O H, O I may be shew'd to be equal. Therefore a Circle describ'd from the Center O, which passeth through D, will also pass through G, H, I, and touch all the Sides of the Square (*b.*) Because the Angles at D, G, H, I are right ones. Therefore we have done what was required.

(a) *Per* 26. *l.* 1.
(b) *Per* 16. *l.* 3.

PROP. X. Problem.

TO make an Isosceles Triangle B A C, in which the Angle at the Base (A B C, or A C B) shall be double to that which is at the Top (A.) Fig. 7.

Let any right Line, what you will, as A B, be taken, which so cut in D (*c*) that the Rectangle A B D shall be equal to A D Square. Then from the Center A, thro' B, describe a Circle; in which inscribe (*d*) B C equal to A D, and join A C. B A C shall be the Triangle sought.

(c) *Per* 11. *l.* 2.
(d) *Per* 1. *l.* 4.

For let the right Line D C be drawn, and through A, D, C describe (*e*) a Circle. Because the Rectangle A B D is equal to the Square A D, (that is, B C) it is manifest, that B C (*f*) toucheth the Circle D O, which C D cuts. Therefore the Angle B C D (*g*) is equal to the Angle A in the opposite Segment; and so the common Angle D C A being added, B C A must be equal to A+D C A. But because the Sides A B, A C are equal, A B C (*h*) is equal to the Angle ACB. Therefore the Angle A B C is also equal to A+D C A. But the external Angle also B D C is equal to the two internal ones (*i*) A+D C A. Therefore A B C, and B D C are equal. Therefore the Line D C is (*k*) equal to B C, (that is, by the Construction to D A.) Therefore the Angles A and D C A (*l*) are equal. Wherefore the Angle A B C, which hath been shew'd equal to those two, shall be double to one A. That is done therefore which was required.

(e) *Per* 5. *l.* 4.
(f) *Per* 37. *l.* 3.
(g) *Per* 32. *l.* 3.
(h) *Per* 5. *l.* 1.
(i) *Per* 32. *l.* 1.
(k) *Per* 6. *l.* 1.
(l) *Per* 5. *l.* 1.

Corollary.

Corollary.

EACH of the Angles at the Bafe B and C in the *Ifofceles* now framed, is two Fifths of two right ones, or four Fifths of one right one, and the remaining one A is one Fifth of two right ones, or two Fifths of one right one. And is manifeft out of this Propofition taken together with that 32. l. 1.

PROP. XI. Problem.

Fig. 7, 8. TO *infcribe a regular Pentagon in a Circle.*

(a) *By the foregoing.* (b) *Per* 2. *l.* 4. Let there be defcribed (*a*) the Triangle B A C, having the Angles at the Bafe double to that at the Top. Infcribe a Triangle C A D equi-angled to this in a Circle (*b.*) Bifect the Angles at the Bafe A C D, A D C, with the right Lines C E, D B, cutting the Circle in E and B. The Points A, B, C, D, E, join'd by right Lines, will give an ordinate Pentagon infcrib'd in a Circle.

(c) *Per* 28. *l.* 3. (d) *Per* 27. *l.* 3. (e) *Per* 29. *l.* 3. For from the Conftruction it appears that the Angles I, N, Q, S, O are equal. Wherefore the Arches fubtended to them, A E, E D, C D, C B, B A, are alfo (*c*) equal. Therefore the right Lines fubtended to thofe Arches, fhall alfo (*d*) be equal. The Pentagon therefore is Equilateral. But it is alfo (*e*) Equiangular, becaufe its Angles B A E, A E D, *&c.* ftand on equal Arches B C D E, A B C D, *&c.* that therefore is done which was required.

Corollary.

Fig. 8. THE Angle of a regular Pentagon makes fix Fifths of one right Angle, or three Fifths of two. For the three Angles at A, feeing they are equal, as ftanding upon equal Arches, B C, C D, D E, and the middlemoft of them by the Corollary foregoing is two Fifths of one right Angle; the three together, that is, the Angle of the Pentagon itfelf muft make fix Fifths of one right one.

[Scholium.

[Scholium. " This holds universally, that Figures of an *Fig.* 8.
" odd Number of Sides are inscribed in a Circle, by means
" of an *Isosceles* Triangle, whose equal Angles at the Base
" are multiple of those at the Top. But Figures of an
" even Number of Sides are inscribed by the means of
" *Isosceles* Triangles, whose Angles at the Base are each of
" them multiple sesquialteral of that which is at the Top.
" As in the *Isosceles* A C D, if the Angle C or D be
" threefold of A, the Side C D will be the Side of an
" Heptagon ; if fourfold, it will be the Side of an Enne-
" agon, &c. But if C or D shall be one and a half of A, CD
" will be the Side of a Square ; and if C shall be two and
" a half of the Angle A, C D will subtend a sixth Part of
" the Circumference : In like manner, if C or D shall be
" three and a half of the Angle A, C D shall be the Side of
" an Octagon, &c.

Scholium.

EUCLID's Inscription of a Pentagon is ingenious, but that of *Ptolomy*, which he delivers in the first Book of his *Almagest*, is much more expeditious : And it is this :

Let the Diameters E D, B F be drawn, cutting one another perpendicularly in A. Bisect the Radius A D in C. From the Center C, through B, describe an Arch, meeting the Diameter E D in G. The right Line G B is the Side of a Pentagon, and A G of a Decagon. *Fig.* 12.

The Demonstration cannot be given here, for it depends upon the 13th Book of *Euclid.* See it in *Clavius*, in his Scholium, after *Prop.* 10. *l.* 13.

Problem.

UPON a given right Line (A B) to describe a regular Pentagon. *Fig.* 9.

Cut AB so in C (*a*) that the Rectangle A B C may be equal to the Square of A D. From A B protracted on both sides take away A D, B E equal to the greater Segment A C. From the Centers A and D, with the Interval A B, describe two Arches, cutting each other in F. Likewise, from the Centers B and E, describe, with the same Interval,

(a) *Per* 11. *l.* 2.

val, two Arches cutting each other in G. And again, from the Centers G and F, with the ſame Interval, deſcribe two others, cutting each other in I. The Points A, F, I, G, B, being join'd, will give a regular Polygon upon the right Line A B.

That it is Equilateral, is manifeſt from the Conſtruction; that it is aqui-angled, will be thus demonſtrated. Let D F be drawn. It is manifeſt by the Conſtruction, that A D F is an *Iſoſceles.* And the Baſe A D is the greater Segment of the Side D F, ſo divided, that the Rectangle of the whole and the leſſer Side, is equal to the Square of the greater (For D F is equal to A B, and A D equal to A C.) Therefore the Angle D A F is two Fifths of two right ones; by *Corol. Prop.* 10. *l.* 4. Therefore the remaining Angle F A B is three Fifths of two right ones, or ſix Fifths of one right

(b) *Per* 13. *l.* 1. one (*b*); and therefore is an Angle of a regular Pentagon (*c*). In the ſame manner may it be ſhewn, that the Angle G B A

(c) *Per Corol.* 5. *p.* 11. *l.* 4. is three Fifths of two right ones, and ſo equal to F A B. From whence it is neceſſary, that the reſt F, G, I, ſhould be equal to theſe, as appears from their being Equilateral to theſe, if the right Line F G be conceiv'd to be ſubtended, as appears by *Prop.* 8. *l.* 1.

PROP. XII. Problem.

Fig. 10. *TO circumſcribe an ordinate Pentagon about a Circle.*

Let there, by the foregoing, be inſcrib'd the regular Pentagon G H I K M, and let there be drawn Tangents in the Points G, H, I, K, M, which may concur in B, C, D, E, F. I ſay the Thing is done.

For from the Center draw the right Lines, A G, A B, A H, A C, A I. Here, becauſe from the ſame Point B,

(a) *Per Corol.* 2. *p.* 36. *l.* 3. B G, and B H touch the Circle, they (*a*) are equal. Therefore the Triangles G A B, B A H are Equilateral to each other. Therefore (*b*) the Angles O, P, as likewiſe

(b) *Per* 8. *l.* 1. thoſe Q, S, are equal. And therefore the whole Angle B is double to P, and the whole G A H double to S. For the ſame reaſon the Angles C and H A I are double to T and N reſpectively. But G A H and H A I are equal (c)

(c) *Per* 29. *l.* 3. becauſe they ſtand upon equal Arches, by Conſtruction G H, H I. Therefore their halves, S and N are also

equal.

equal. Becaufe therefore in the Triangles B A H, H AC, the two Angles S and N are equal, and thofe at H are both right Angles (*d*,) and likewife the Side A H is common; therefore the Sides (*e*) B H, C H, as likewife the Angles P, T, are equal. In the fame manner I might fhew B G, F G to be equal. Therefore B F, C B, which are double to the Equals B G, B H, are alfo equal. In the fame manner it may be fhew'd that the reft of the Sides of the circumfcribed Pentagon are equal. It is therefore Equilateral; but it is alfo equi-angled; for feeing it hath been fhew'd that the Angles B and C are each of them double to the Equals P and T they muft alfo be equal betwixt themfelves. And in the fame manner of the reft We have therefore defcribed a regular Pentagon about a Circle. Which was the Thing to be done.

(d) *Per* 18. *l.* 3.

(e) *Per* 26. *l.* 1.

In the fame way any ordinate Figure whatfoever is defcrib'd about a Circle, that is, if a like Figure be firft infcrib'd in the Circle.

PROP. XIII, XIV. Problems.

To infcribe a Circle in a regular Pentagon, and circumfcribe one about it. *Fig.* 11.

Bifect the two Angles of the Pentagon B, C, with the right Lines B N, C S, cutting each other in A. From A draw the Perpendicular A L.

A Circle defcrib'd from the Point A, with the Interval A L, touches all the Sides of the Pentagon; and a Circle defcrib'd from the fame Point A, with the Interval A B, paffes alfo through the Points E, F, D, C.

Part I. In the Triangles D C A, B C A, becaufe the Sides D C, CA, are equal to B C, C A, by the Hypothefis, and the Angles P and O are equal by the Conftruction, thofe alfo G and I will be equal by 4. *l.* 1. Now, the whole alfo B and D are equal by the Hypothefis. Wherefore feeing the Angle G is half of B by the Conftruction, I alfo will be half of D. Therefore D is bifected by the right Line D M. For the fame Caufe the reft of the Angles of the Pentagon E F, are bifected, and confequently all the half Angles are equal among themfelves. Now, let the Perpendiculars be drawn, A M, A S, A N, A R. For

becaufe

because in the Triangles L B A, M B A, the Angles G and B L A are equal to the Angles Q and B M A, by the Construction, and the Side B A is common, A L and A M must
(a) *Per* 26. *l.* 1. be also equal (*a*.) In like manner I might shew that the rest of the Perpendiculars, A M, A N, A S, A R, are equal. A Circle therefore from the Center A, passing through L, will likewise pass through M, S, N, R, and because the Angles at L, M, S, N, R, are right ones by the Con-
* *Per* 16. *l.* 3. struction, * it will touch the five Sides of the Pentagon. Which was the first Part.

Part II. In the Triangle C A B, because the Angles O and G have already been shewn to be equal, the Sides also
(b) *Per* 16. *l.* 1. A C, A B must be equal (*b*,) and in the same manner, A B, A F, A E, A D, may be prov'd equal, and therefore a Circle from the Center A passing through B, must pass also through C, D, E, F. Therefore we have both inscrib'd a Circle in a Pentagon, and circumscrib'd one about a Pentagon. *Q. E. D.*

[" In the same way, in any regular Figure whatsoever, a Circle may be inscrib'd, and circumscrib'd about it.

PROP. XV. Problem.

Fig. 13. *IN a given Circle to describe a regular Hexagon*

Let the Diameter F A B be drawn. From the Center B through A, describe a Circle, cutting the given one in C and D. Likewise from the Center F, through A, a Circl cutting the given one in E and G. The six Points, B, C E, F, G, D, connected by right Lines, will give the Hexagon required.

From the Center A, let fall the right Lines A E, A C A G, A D. It is manifest that the Triangles H, I, M L, are Equilateral, both in themselves, and with one ano-
(c) *Per* 1. *l.* 1. ther (*c*.) Then because the Angles C A B, E A F, eac of them make one Third of two right Angles (*per Corollar* 12 *p.* 32. *l* 1.) and therefore do make both together tw
(d) *Per Corol.* 1. *p.* 13. *l.* 1. Thirds of two right Angles; it remains (*d*) that E A C one Third of two right Angles; therefore the Angl E A C, C A B are equal. But the Sides also E A, A C are equal to the Sides B A, A C. Therefore the Ba

E

fore the Base E C (*per* 4. *l.* 1.) is equal to the Base B C, that is, to the Radius A C by the Construction. Wherefore the Triangle N is also Equilateral And in the same manner the Triangle K may be shewn to be so. Because therefore all the six Triangles H, I, K, L, M, N, are Equilateral; it is manifest that all the Sides, C B, B D, D G, G F, F E, E C, are equal one to another, and to the Radius A C. The Hexagon is therefore Equilateral. But it is also Equiangular, seeing each one of its Angles E C, B D, G F, consists of two Angles of an Equilateral Triangle. Therefore we have inscribed a regular Hexagon in the Circle.

Corollaries.

1. THE Side of an Hexagon inscrib'd in a Circle, [or a Chord of 60 Degrees] is equal to the Radius, [and consequently the Sine of 30 Degrees is equal to half the Radius, *per Corollary* 2 *p.* 3. *l.* 3.]

2. An Angle of a regular Hexagon is four Thirds of one right Angle; as consisting of two Angles of an Equilateral Triangle, each of which makes two Thirds of a right Angle.

3. If there be drawn the Diameter P S, perpendicular to the other F B; and with the Interval of the Radius P A, from the Center P and S, there may be made Sections in O and Q, in R and T, and in like manner from the Centers F and B, make the Sections in G and E, in D and C, the Points, P, E, O, F, R, G, S, D, T, B, Q, C, connected with right Lines, will give a Figure of twelve Sides, inscrib'd in a Circle with one Aperture of the Compasses. Which Thing is of great Service in Dialling. *Fig.* 14.

4. From what has been demonstrated, we may easily describe an Equilateral Triangle in a Circle. The Diameter FB being drawn, from the Center B, through A, describe the Arch C A D. The Points C, F, D, connected with right Lines, will give the Triangle sought. *Fig.* 14.

5. The Side C X D of the Equilateral Triangle, cuts off from the Diameter B F perpendicular to it, a fourth Part thereof B X For the Angles A C X, B C X, standing upon equal Arches G D, D B are equal (*per* 29. *l.* 3.) and the Sides, A C, C X, are equal to the Sides B C, C X. Therefore A X, B X are equal (*a.*) Therefore B X is the fourth Part of the Diameter B F.

(a) *Per* 4. *l.* 1.

Scholium 1. *Problem.*

Fig. 13. * *Per* 1. *l.* 1.

YOU may raiſe a regular Hexagon upon a right Line BC thus. Make an * Equilateral Triangle CAB upon the given Line CB. From the Center A through B and C deſcribe a Circle. This will contain an Hexagon upon the given right Line CB. The Thing is manifeſt from the Propoſition, and *Corollary* 1.

Theorem.

THE Square of a Side of an Equilateral Triangle is treble to the Square of the Semi-diameter of a Circle in which it is inſcrib'd, and is to the Square of the whole Diameter, as 3 to 4.

Fig. 14.

Let there be drawn the Semi-diameter A D. The Square of F D is equal to F A q + D A q + the Rectangle F A X twice taken (*per* 12. *l.* 2. But the Rectangle F A X twice taken, is equal to the Square of the Semi-diameter F A or DA: (For becauſe A X, X B (*b*) are equal, the Rectangle FAX twice taken, is equal to the two Rectangles which are under FA, AX, and under F A and XB, that is, equal to the Rectangle FAB (*c*); that is, equal to F A q.) Therefore F D q is treble to F A q or D A q, the Square of the Semi-diameter.

(b) *Per Corol.* 5. *foregoing.* (c) *Per* 1. *l.* 2.

Now, becauſe the Square of the whole Diameter is Quadruple of the Square of F A, the Semi-diameter (*d*), it is manifeſt that the Square of F D is to the Square of the Diameter, as 3 to 4.

(d) *Per Corol.* 3. *Prop.* 4. *l.* 2.

Hence it follows, that a Side of an Equilateral Triangle is to the Diameter, as the Square Root of 3 is to 2, the Square Root of 4; and therefore that thoſe Lines are incommenſurable.

PROP.

PROP. XVI. Problem.

TO inscribe a regular Quindecagon in a Circle. *Fig.* 15.

Inscribe the Circle A C, the Side of a Pentagon (*a*), and A D, the Side of an Equilateral Triangle (*per* Corol. 4. *p.* 15. *l.* 4. bisect the Arch C D in E. C E is the Side of the Quindecagon, or fifteen-angled Figure sought. (a) *Per* 11. *l.* 4.

For if the whole Circumference be suppos'd to be 15, the Arch AC will be 3, and the Arch A D 5, and therefore the Arch C D 2, and consequently C E 1.

Corollary.

BY this Method innumerable regular Figures may be inscrib'd in a Circle. For if A C, A D, the Sides of two regular Figures be inscrib'd in a Circle, the Difference of the Arches C D will contain so many Sides of a new regular Figure, as are the Units whereby the Denominators of the former differ one from another. But the Denominator of the new Figure is had, if the Denominators of the former be multiplied one by the other. *Fig.* 15.

As if A D be the Side of a Square, and A C of a Decagon, the Difference of the Denominators is 6. Therefore the Arch C D contains 6 Sides of a new Figure. But the new Figure is of 40 Sides. For the Denominators 4 and 10, multiplied one by the other, make 40.

Scholium.

THERE hath not yet been found out the Art by which regular Figures of 7, 9, 11, 13, 17, *&c.* Sides may be inscribed in a Circle, by a Pair of Compasses and a Rule only; forasmuch as that Inscription of Figures depends upon the Division of the Circumference into any given Parts, which thing is lacking: But if the Circumference of a Circle be divided into 360 Parts, you may, in a mechanical Way, inscribe any regular Figure whatsoever, in it, after this manner.

Problem 1.

Fig. 15. DIVIDE 360 Degrees (that is, the whole Circumference, by the Denominator of the Polygon to be inſcrib'd (*e. g:* a Nonangle.) Make at the Center the Angle A G K of ſo many Degrees as are the Units of the Quotient in the ſaid Diviſion. A K ſhall be the Side of the nine-angled Figure, which is required to be inſcrib'd in the Circle.

Problem 2.

Fig. 15. BUT upon a given right Line you may deſcribe any regular Figure whatſoever by the help of the following Table.

A right Angle is to the Angle of the Figure,

				Difference.
In a Pentagon, as	5	to	6	—1
In an Hexagon, as	3	to	4	—1
In an Heptagon, as	7	to	10	—3
In an Octagon, as	2	to	3	—1
In a Nonagon, as	9	to	14	—5
In a Decagon, as	5	to	8	—3
In an Undecagon, as	11	to	18	—7
In a Duodecagon, as	3	to	5	—2

Fig. 15. Let a regular Heptagon be to be inſcrib'd upon the given right Line X B. From the Center X, with the Interval X B, deſcribe a Circle, from which cut off the Quadrant BO. See in the Table what is the Proportion of a right Angle, to the Angle of an Heptagon: You will find it to be as 7 to 10, and the Difference is 3. Divide the Quadrant therefore into ſeven equal Arches, ſo many of which add to it from O to N, as the Difference hath Units. Through three Points, B, X, N, deſcribe (*per* 5. *l.* 4.) a Circle. This contains an Heptagon of the given right Line X B.

The Table was made by means of *Theorem* II. in the *Schol.* upon *p.* 32. *l.* 1. by which is found the Number of right Angles, which the Angles of any right-lin'd Figure make; which Number being divided by the Denominator of the Figure, gives the Denominator of the Proportion of the Angle of the Figure to a right one.

Now,

Now, becauſe hitherto many Things have been propounded concerning regular Figures, let the following famous Theorem of *Proclus* cloſe this Book.

Theorem.

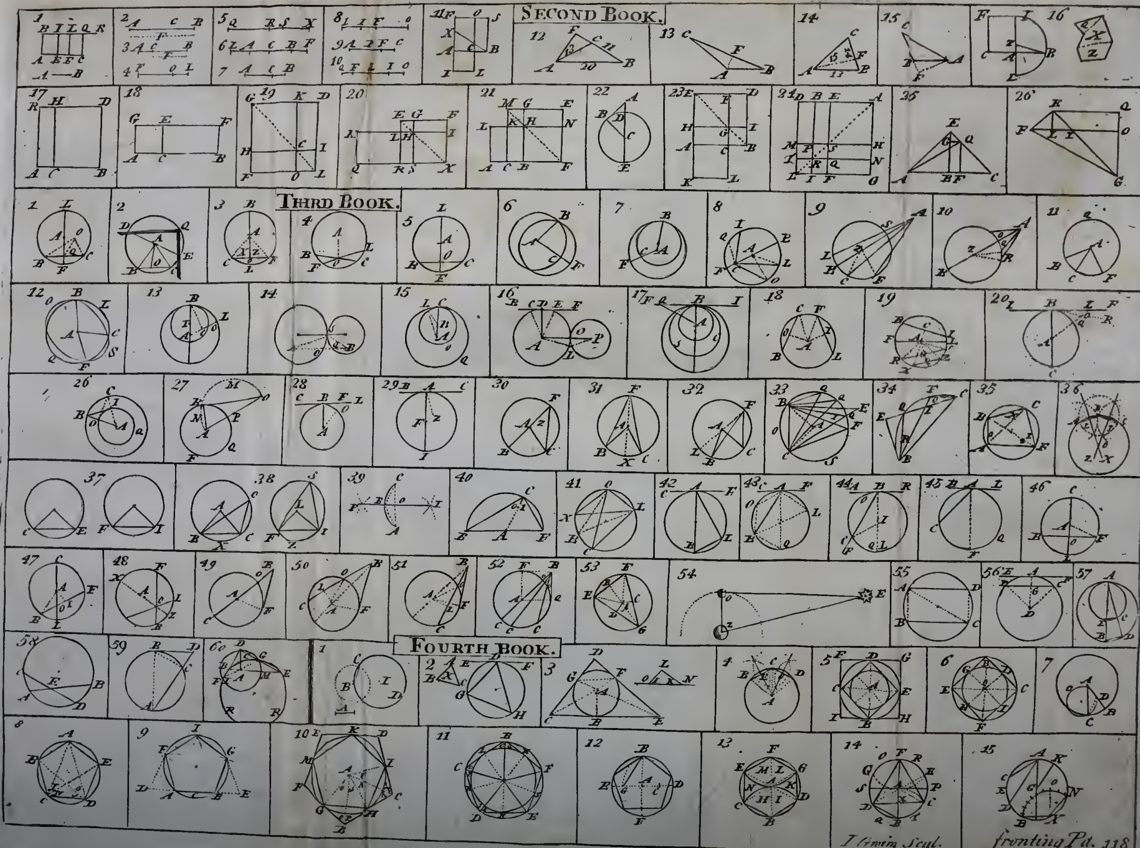
ONLY three regular Figures, to wit, ſix Equilateral Triangles, four Squares, and three Hexagons, can fill a Space; that is, conſtitute one continued Superficies. Which is thus demonſtrated. That ſome regular Figure, often repeated, ſhould be able to fill a Space; it is required that the Angles of many Figures of that kind being diſpoſed about one Point, ſhould make juſt four right ones; for juſt ſo many right Angles may be placed about one Point, as appears from *Corollary* 3. *Prop.* 13. *l.* 1. As for Example; that Equilateral Triangles ſhould fill a Space, it is requir'd that ſo many Angles of ſuch Triangles N, M, L, K, I, H, being diſpos'd about the Point A, ſhould make juſt four right ones But ſix Angles of an Equilateral Triangle do make four right ones;) for one makes two Thirds of one right one *, and therefore ſix of them make twelve Thirds of one right one, that is, four right ones:) Likewiſe the four Angles of a Square make four right ones, as is manifeſt; likewiſe three Angles of an Hexagon; for one maketh four Thirds of one right Angle, *(per Corollary* 2. *p.* 15. *l.* 4.*)*; and therefore three of them do make twelve Thirds of one right Angle, that is, four right ones. Therefore, &c.

But that no other Figure beſides theſe can do this, will manifeſtly appear, if its Angle being found, as above, you ſhall multiply the ſame by any Number whatſoever; for the Angles will always either fall ſhort of, or exceed four right ones.

THE
Elements of EUCLID.

BOOK V.

THIS fifth Book of Elements is altogether neceſſary for demonſtrating the Propoſitions of the ſixth Book. The Doctrine which it containeth is almoſt in continual Uſe. The Way of Reaſoning from Geometrical Proportion is moſt ſubtil, ſolid and brief. This Method of Reaſoning, as a kind of Mathematical Logick, Geometry, Arithmetick, Muſick, Aſtronomy, Staticks, and all the other Parts of the Mathematicks, make eſpecial Uſe of: Foraſmuch as they almoſt wholly depend upon Proportions connected together one with another; and are wont to borrow their Ways of Reaſoning concerning Proportionals from this fifth Book. Practical Geometry, which conſiſts in the meaſuring of Lines, Figures and Solids, is for the moſt part derived from the Doctrine of Proportions. There is not a Rule in Arithmetick, but what may be demonſtrated from the Propoſitions of this fifth Book, without the help of the 7th, 8th and 9th Books, which treat profeſſedly of Numbers. We may fitly call the Muſick of the Antients, Geometrical Proportions apply'd to tuneful Sounds; which ſame Thing you may well nigh ſay concerning Staticks, which are converſant about the Weights of Bodies. To comprehend the whole Matter in few Words; If you take away the Doctrine of Proportion from the Mathematicks, you will leave almoſt nothing which is excellent, or greatly to be accounted of.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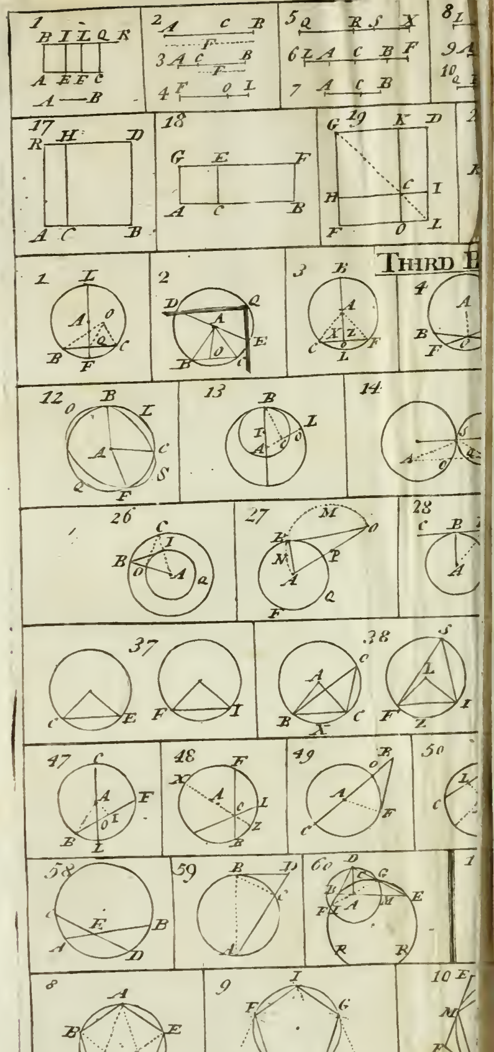

THIRD

Scholium.

" There is no Mathematician who is ignorant of how " great Importance in Geometry the Knowledge of Pro- " portions is; for it is the very Marrow, as it were, of the " Mathematical Sciences: And the various Ways of Rea- " ſoning concerning Proportionals, are both moſt uſeful, " and moſt certain; neither can we without them move " one Step.

" But then I reckon that this Doctrine is congenite in " Men's Minds with common Reaſon itſelf; and that the " various Ways of Reaſoning concerning Proportionals, " which *Euclid*, by much winding and going about, de- " livers in this whole Book, do not ſo much need Demon- " ſtration, properly ſo call'd, as Illuſtration and Examples. " And I am altogether of Opinion, that thoſe who take in " hand to deliver this moſt eaſy Doctrine by a long Circuit " of Propoſitions, do involve a Thing in it ſelf moſt clear, " in a certain Cloud, and render it far more difficult. The " Sum of the Matter I will open in a few Words. It is a " thing eaſily known, that four Quantities are then pro- " portional, or that the Analogies are then alike, when the " firſt Quantity contains the ſecond, as often as the third " contains the fourth; or when the firſt is as often con- " tain'd by the ſecond, as the third is by the fourth. So " 16 : 8 :: 4 : 2. And 3 : 9 :: 4 : 12. Here are like, " or the ſame Proportions; becauſe in the former Exam- " ple, the Conſequents 8 and 2 are contain'd twice in their " reſpective Antecedents; and ſo the Proportion of the An- " tecedents to the Conſequents is double. And in the " other Example. the Proportions are alſo alike, becauſe " the Conſequents 9 and 12 do contain their reſpective An- " tecedents three times; and ſo the Proportion of the An- " tecedents, to the Conſequents, is ſub-triple. (Nor is " there any Proportion of commenſurable Quantities " which may not be expreſs'd by certain Numbers; nor " indeed of Incommenſurables, which may not be expreſſed " by Numbers infinitely approaching nearer and nearer " unto the true one.) Furthermore, from what hath " been ſaid it appears, that like Proportions, whatſoever " they are, may be expreſſed not only by divers Numbers, " but alſo by the ſame. Thus 2 to 1 deſigns as well the

" Proportion of 16 to 8, as of 4 to 2; 1 to 3 no less ex-
" presseth that of 4 to 12, than that of 3 to 9, as is
" most manifest. Supposing therefore four Quantities to
" be proportional, A : B : : a : b; it is enquir'd in this
" Book after how many like Manners the Terms of these
" like Proportions may be changed, and ordered amongst
" themselves So that the emerging Proportion on both
" Sides may be still alike. And it may be answered, that
" it may be done after all the Ways and Manners possible;
" for seeing the Proportion of A to B, and that of a to
" b are alike, both of them may be expressed by the
" same Numbers after this manner, A : B : : 9 : 3, and
" a : b : : 9 : 3. And consequently all the Proportions
" emerging on both Sides, either by *Alternating* the
" Terms, or by *Inverting* them, or by *Compounding*,
" or *Dividing*, or *Converting*, or *Mixing* them, may
" be expressed by the *very same* Numbers; and conse-
" quently the same Proportion will always be kept on
" both Sides. As for Example sake. A + B : B : : a + b
" : b, because 9 + 3 : 3, expresseth the same Propor-
" tion; which is *Composition*. The same is to be said of
" all the Ways of changing the Terms Therefore let
" Beginners observe this one Thing, that Proportions,
" which are on both Sides the same, be ever changed and
" ordered in the very same manner. And then there will
" be no room to question, whether the Proportions which
" arise on both Sides be alike or no. It is indeed a Thing
" to be wonder'd at, that no one of those who have hither-
" to compiled Elements of Geometry, have made use of
" this most easy Method of Stating the Equality of Pro-
" portions, for the Illustrating of this Fifth Book about
" the Doctrine of Proportions. Take therefore the pri-
" mary Ways which Geometry makes use of, in Reasoning
" concerning like Proportions, as they are digested into
" this short Table.

Let

Let it be	$A : B :: a : b :: 9 : 3$
Then it will be by	
Alternating	$A : a :: B : b :: 9 : 9 :: 3 : 3$
Inverting	$B : A :: b : a :: 3 : 9$
Compounding	$A+B : B :: a+b : b :: 9+3\ (12) : 3.$
Dividing	$A-B : B :: a-b : b :: 9-3\ (6) : 3$
Converting	$A : A+B :: a : a+b :: 9 : 9+3\ (12)$
or	$A : A-B :: a : a-b :: 9 : 9-3\ (6)$
Mixing	$A+B :: A-B :: a+b : a-b :: 9+3 : 9-3:$
Ex æquo	$A : B :: a : b$, & $B : C :: b : c$, *then* $A : C :: a : c.$
	$9 : 3 :: 9 : 3$, $3 : 1 :: 3 : 1$, *then* $9 : 1 :: 9 : 1.$
Ex æquo	$A : B :: a : b$, & $B : C :: r : a$, *then* $A : C :: r : b.$
perturbate.	$8 : 3 :: 8 : 3$, & $3 : 12 :: 2 : 8$, *then* $8 : 12 :: 2 : 3.$
Or thus,	$8 : 3 :: 16 : 6$, & $3 : 2 :: 24 : 16$. *then* $8 : 2 :: 24 : 6.$
	$a : b :: e+a : a+b$ & $b : e :: a+b : a+e$ *then* $a : e :: a+b : e+b.$

" He therefore that is expert in these Ways of Reasoning " concerning Proportionals, and knows how to bring them " into Use upon occasion, will seldom stand in need of the " particular Propositions of the Fifth Book. Only two of " them, which yet are almost Axioms, may not improperly " be inserted and illustrated by Example, in Way of *Appen-* " *dix*, because of the Frequency of their Use in all Parts " of the Mathematicks ; which therefore shall be done after " the Definitions.

DEFINITIONS.

1. AN *Aliquot* Part of Magnitude, is that which being so many times more or less repeated, doth measure, or is just equal to the Magnitude. An *Aliquant* Part is that which doth not measure it.

The Length of one Foot is an *Aliquot* Part of the Length of 10 Feet, because being ten times repeated, it measures it. But the Length of 4 Feet is an *Aliquant* Part of a Line of 10 Feet, because being so many times repeated, to wit, twice, it falls short of it, but being thrice repeated, it exceeds it.

2. One Magnitude is said to be a *Multiple* of another, when the lesser measures the greater, and consequently is an

as *Aliquot* Part thereof; or when the greater contains the lesser so many times precisely.

3. Proportion is the mutual Respect, as to Quantity, of two Magnitudes of the same Kind.

Therefore there are in all Proportions two Terms, of which that is called the Antecedent which is first named, or which is nam'd in the Nominative Case; the other the Consequent.

When the Antecedent and the Consequent are equal, it is called Proportion of Equality; when they are unequal, Proportion of Inequality.

4. Rational Proportion is that which is betwixt commensurable Magnitudes, and may be expressed in Numbers Irrational Proportion, is that which is betwixt Quantitie incommensurable, and cannot be explicated by any Numbers.

Moreover, commensurable Quantities are those which some common Measure measureth; Incommensurable, those which cannot be measured by any common Measure.

Fig. 1. *l.* 5. 5. Two Proportions (that of A to B, and that of C to F) are alike, equal or the same; when the Antecedent of one [A] doth equally, or in the same manner, that is neither more nor less, contain its Consequent [B] as the Antecedent of the other [C,] contains its Consequent [F.]

Fig. 2. Or when the Antecedent of the one [A] is so often contain'd in its Consequent [B] as [C] the Antecedent of the other is in its Consequent [D.]

Fig. 3. 6. Two Proportions are unlike, or one is greater than the other, when the Antecedent of one [I] doth more contain its Consequent [L,] than the Antecedent of the other *Fig.* 4. [O] doth contain its Consequent [Q;] or when the Antecedent of one is less contain'd in its Consequent, than the Antecedent of the other in its Consequent.

7. Like or *similar* Parts are those which are equally or in the same manner contain'd in their Wholes; so that what sort of Part one is of its Whole, such a Part the other is of its Whole. Which thing indeed is nothing else, but that the Parts bear the same Proportion to their Wholes.

Aliquot Parts are alike, which do equally measure their Wholes, as if each of them be one Third or one Tenth &c. of its Whole.

8. Magnitudes [A, B, C. D] are said to be *continual* proportional when the middle Terms [B, C] are taken twice

twice; that is, when they are each of them a Consequent in respect of the foregoing, and an Antecedent in respect of the following. Fig. 6.

We thus pronounce continual Proportions. A is to B, as B to C; and B is to C, as C is to D. And so on.

9. Magnitudes are *discretely* proportional when no Term is twice taken.

Discrete Proportions we thus pronounce: A is to B, as C to F. When they are more than three proportional Magnitudes, if they be said to be proportional, they are always understood to be *discretely* so. Fig. 1.

10. When the Magnitudes [A. B. C, D] are continually proportional, the first [A] is said to have to the third [C] a duplicate Proportion of that which it hath to the second [B :] And the first [A] is said to have to the fourth [D] a triplicate Proportion of that which the same first hath to the second [B :] And so forwards. Fig. 6.

[" If one triplicate Proportion be equal to another dupli-
" cate Proportion, the latter simple Proportion shall be
" sesquiplicate, or one and a half of the former simple
" Proportion Let A, B, C, D, be ∺; and a, b, c, ∺;
" and let A the first in the former Analogy be unto D the
" fourth; as [a] the first in the second Analogy is to [c]
" the third; I say, that [a] is to [b] in a Proportion,
" which is one and a half of that which A is in to B.
" For let F be a middle Proportional betwixt B and C:
" Or, which is the same thing, betwixt A and D Be-
" cause of the Equality of the Proportions of A to D, and
" [a] to [c] and the middle Proportions on both Sides F
" and [b ;] it will be A : F : : a : b. But the Proportion
" of A to F is compounded of the entire Proportion of
" A to B, and of the Proportion of the same B to F
" halved; and consequently the Proportion of [a] to
" [b] which is equal to that of A to F, contains the
" entire Proportion of A to B, and also the same halv'd,
" to wit, the Proportion of B to F. But the whole Pro-
" portion, with its half, is a sesquiplicate or sesquialteral
" Proportion, or that which is one and a half of the
" other. [a] Therefore is to [b] in a Proportion sesqui-
" plicate of that of A to B. So in Astronomy, since the
" Cubes of the Distances of the Planets from the Sun bear
" that Proportion one to another, which the Squares of

" their

" their periodical Times bear ; ſo that the triplicated Pro-
" portion of the Diſtances, is the ſame with the Dupli-
" cate one of the periodical Times ; it is wont to be ſaid,
" that the periodical Times are in a ſeſquiplicate or ſeſqui-
" alteral Proportion of their Diſtances from the Sun.]

Fig. 1. 11. Antecedent Magnitudes are ſaid to be Homologous, or like to Antecedent, and Conſequent to Conſequent Magnitudes. As if A is to B, as C to F ; A, C, and B F, are homologous Quantities.

XII. If a Set of Pairs of Quantities contain every one the ſame Proportion, that is the very Proportion alſo which the Sum of all the Antecedents bears to the Sum of all the Conſequents.

$$\begin{array}{c} 20+6+8+18+14=66 \\ \overline{10+3+4+9+7=33} \end{array}$$

XIX. If Parts be as Wholes, the Remainders will be alſo in the very ſame Proportion.

If 30 be to 20, as 3 to 2 ; 27 will be to 18 alſo as 30 to 20, or as 3 to 2.

T H I

THE

Elements of EUCLID.

BOOK VI.

THE Doctrine of Proportions, which was generally ſet forth in the Fifth Book, is applied in the Sixth, to plain Figures. And thoſe Things which are delivered in this Book are ſo neceſſary to be known, that without them no Man can penetrate into the Secrets of Geometry, and reap the ſweet Fruits of the Mathematicks. Each Propoſition deſerves to have an Encomium annexed; ſo great is the Utility of all.

" This Sixth Book, as hath been ſaid, begins to apply " that excellent Doctrine concerning Geometrical Propor- " tion, which was juſt before delivered, to divers, and " thoſe certainly, moſt notable Uſes; and beginning with " Triangles, the moſt ſimple of Figures, ſearches out their " Sides and Areas, as they anſwer to one another in a cer- " tain Proportion. Then it defines proportional Lines, and " the proportional Augmentations, or Diminutions of Fi- " gures; and ſhews in what manner we may either encreaſe " or diminiſh them according to any Proportion given. It " opens likewiſe the Golden Rule, or Rule of Proportion, " the very chief of all Arithmetick; and demonſtrates " that in a Rectangle Triangle, not only the Square, but " alſo the Pentagon, Hexagon, and in general, every re- " gular Polygon, which is deſcribed by the Hypothenuſe, " is equal to the Squares, Pentagons, Hexagons, or any " regular Polygons whatſoever, that are deſcrib'd by the

" two

" two Sides. It alſo propounds moſt eaſy and certain
" Principles for meaſuring as well Solids, as Lines and Sur-
" faces, which are of very great Uſe in all Parts of the
" Mathematicks.

DEFINITIONS.

1. LIKE or ſimilar Figures, are thoſe which both have all the Angles equal, each to each other reſpectively, and the Sides which are oppoſed to the equal Angles, or which are betwixt them, or which are about the equal Angles, (for they come all to one) Proportional.

Fig. 7. *l.* 6. As the Triangles X, Z, will be ſaid to be like, or ſimilar, if the Angle A be equal to the Angle F, and the Angle B equal to the Angle I, and conſequently the Angle C equal to the Angle L: And moreover, if A B be to F I, as B C to LI; and BC is to LI, as CA is to L F; and CA to L F, as AB to F I; by comparing always the Sides oppoſite to the equal Angles. In the ſame manner the Likeneſs of all right-lin'd Figures may be explained

Fig. 29 2. Reciprocal Figures are when the Antecedent and Conſequent Terms of the Proportions appear on both Sides.

As in the Parallelograms X Z,
If A C be to C B,
As F C is to C L.

The Antecedents here are AC and F C, of which there is one in both Figures; and the Conſequents are C B, and C L; of which likewiſe there is one in each Figure. And therefore the Parallelogram X, Z, are called Reciprocal: Underſtanding the ſame of other Figures.

3. The Altitude of a Figure is the Perpendicular let fall from the Top to the Baſe. This is with *Euclid* the fourth Definition.

Fig. 1. A. the Altitude of the Triangle A B C is the Perpendicular A Q, which falls from the Top upon the Baſe B C, either within the Triangle or without, upon the Baſe protracted Now the Baſe and Top are aſſumed at Pleaſure.

4. Like Arches of Circles are thoſe which have the ſame Proportion unto their whole Circumferences.

As if each of them be a Third or Fourth Part, &c. of that Circumference.

PRO

PROPOSITION I. Theorem.

TRiangles (ABC, DEF) and Parallelograms (AOPC, DQRF) which are betwixt the same Parallels, or have the same Altitude, have the same Proportion betwixt themselves as their Bases, (AC, DF) Fig. 2.

Upon this Theorem the whole Sixth Book depends, yea, whatsoever any where is demonstrated about Figures by Proportions, whether Plane or Solid.

Let there be taken any Aliquot Part of the Base DF; e. g. DG one Third, and let the right Line GE be drawn: The Triangle DEG will likewise be one Third Part of the Triangle DEF, as is gathered from 38. l. 1. Wherefore DG, and the Triangle DGE are like Aliquot Parts of their Consequents*. Then let there be taken away DG from the Base AC, as often as it can, as suppose six times, and let the right Lines HB, IB, KB, LB, MB, NB, be drawn. Because the Lines CH, HI, *&c.* are each of them equal to DG, the six Triangles CBH, HBI, *&c.* are each of them (*a*) equal to the Triangle DEG. Therefore as often as DG is contain'd in the Antecedent AC, so often is the Triangle DEG contain'd in the Triangle ABC. By the same Reasoning it may be shew'd, that the like Aliquot Parts whatsoever of the Consequents (the Base DF, and the Triangle DEF) are in an equal Number contain'd in the Antecedents (the Base AC, and the Triangle ABC:) Therefore as the Base AC, is to the Base DF; so is the Triangle ABC, to the Triangle DEF. *Q. E. D.*

* *Per Def.* 7. *l.* 5.

(a) *Per* 38. *l.* 1.

But now because the Parallelograms AP, DR are (*b*) double to the Triangles ABC, DEF, they also will be as their Bases.

(b) *Per* 41. *l.* 1.

Corollary.

THE Triangles (ABC, FIL) and the Parallelograms which have equal Bases (AC, FL) or the same, have that Proportion one to another, which their Altitudes (BO, IQ) have. *Fig.* 3.

For

For let Q S, O R, be made equal to the equal Bafes (F L, A C;) Q S, O R will then be equal. Draw S I, R B. If in the Triangles O B R, Q I S, you take B O, I Q for the Bafes, O R, Q S, will be their Altitudes; which feeing they are equal, the Triangles O B R, Q I S (*c*) will be betwixt themfelves, as their Bafes B O, I Q. But becaufe by the Conftruction, O R is equal to A C, and Q S equal to F L, the Triangles O B R, Q I S, are (*d*) equal to the Triangles A B C, F I L. Therefore the Triangles A B C, F I L, are alfo as B O is to Q I.

(c) *Per* 1. *l.* 6.
(d) *Per* 38. *l.* 1.
Fig. 50.

Corollary (2.) " Hence a Trapezium A B C D, whofe " Sides A D and B C are parallel, may be divided into any " equal Parts whatfoever. For let C E be made equal " to A D. Becaufe of the Equality of the Angles verti- " cally oppofite (*e*) A F D, E F C, and of the alternate " Angles (*f*) D A F, F E C, and A D F, E C F, and the " Equality of the Bafes A D, C E, by Conftruction, the " Triangles A D F, F C E (*g*,) are equal; and therefore the " Triangle A B E is equal to the Trapezium A B C D. " Therefore the Bafe B E being divided into any equal Parts " whatfoever; as for inftance, three, B G, G R, R E, " the Triangles A B G, A G R, A R E, fhall each of them " be one Third Part of the Trapezium. *Q. E. I.*

(e) *Per* 15. *l.* 1.
(f) *Per* 27. *l.* 1.
(g) *Per* 26. *l.* 1.

PROP. II. Theorem.

Fig. 4.

IF to one Side of a Triangle (as BC) there be drawn (FL) a Parallel, this cuts the Sides proportionally, that is, (AF) will be to (FB) as (AL) to (LC.)

And if the right Line (FL) cuts the Sides (BA, CA) proportionally, it will be parallel to the other Side (BC.)

Part I. Let B L, C F be drawn, becaufe F L is fuppofed parallel to B C, the Triangles F B L, L C F having the fame Bafe are (*a*) equal. Therefore the Triangle X hath the fame Proportion to both; now the Triangle X is to the Triangle F B L, as the fame Triangle X is to the L C F. But the Triangle X is to the Triangle F B L (*b*,) as A F is to F B; and the Triangle X is to that L C as A L (*c*) is to L C. Therefore alfo A F is to F B, as A L to L C. *Q. E. D.*

(a) *Per* 37. *l.* 1.
(b) *By the foregoing.*
(c) *By the fame.*

Pa

Part II. As A F is to F B, so is the (*d*) Triangle X to the Triangle F B L: And as A L is to L C, so is the same Triangle X to the Triangle L C F. Now A F is suppos'd to be to F B, as A L is to L C. Therefore the Triangle X is to the Triangle F B L, as the same X is to L C F. Therefore the Triangles F B L, L C F are equal. Therefore seeing they have a common Base F L, the Lines F L, B C, are (*e*) parallel. *Q. E. D.*

(d) *By the foregoing.*

(e) *Per* 39. *l.* 1.

Corollary.

IF unto (B C) one Side of a Triangle there be drawn more Parallels (I O, F L) all the Segments of the Sides will be proportional. *Fig.* 5.

Let F Q be drawn parallel to A C. The right Lines F S, S Q, are equal (*f*) to L O, O C. But B I is to F I, as Q S to S F. (*a*) Therefore B I is also to I F, as C O to O L.

(f) *Per* 34. *l.* 1

(a) *Per* 2. *l.* 6.

PROP. III. Theorem.

IF a right Line (B F) which bisects an Angle of a Triangle, doth also cut the Base (AC,) the Segment of the Base (AF, FC) will have the same Proportion betwixt themselves as the Sides (AB, BC) have. *Fig.* 6.

And if the Parts of the Base (AF, FC) have the same Proportion betwixt themselves, as the other Sides (AB, CB) the Line (BF) which cuts the Base, bisects the opposite Angle (ABC.)

Part I. Draw forth C B until B L be equal to B A; and join A L. Because in the Triangle Z, the Sides L B, A B, are equal, the Angles also (*b*) L and O are equal Because therefore the external Angle A B C is equal to the two internal ones (*c*) L, O, the Angle I, which by the Hypothesis is half A B C, will be equal to the Angle L. Therefore A L, F B (*d*) are parallel. Therefore in the Triangle A C L, A F is to F C (*e*) as L B (that is, A B) is to B C. *Q. E. D.*

(b) *Per* 5. *l.* 1.

(c) *Per* 32. *l.* 1.

(d) *Per* 29. *l.* 1.

(e) *Per* 2. *l.* 6.

Part II. Protract C B again, until B L be equal to B A. Becaufe AF is fuppos'd to be to F C, as A B (that is, L B) is to B C; A L, F B (*a*) are parallel. Therefore the external Angle I is equal to the internal one L (*b*); and the alternate Q equal to the alternate O. But becaufe L B, A B, are equal, the Angles L and O (*c*) are equal. Therefore I and Q are alfo equal. Therefore A B C is bifected. *Q. E. D.*

(a) *Per* 2. *l.* 6.
(b) *Per* 27. *l.* 1.
(c) *Per* 5. *l.* 1.

PROP. IV. Theorem.

TRiangles which are Equiangular to one another, are like or fimilar, that is, have their Sides alfo (a) *that are oppofite to the equal Angles proportional.*

(a) *Per Def.* 1. *l.* 6.

Fig. 7. In the Triangles X, Z, let the Angle A be equal to the Angle F, and the Angle C to the Angle L, and the Angle B to the Angle I; I fay, that A B is to F I, as A C is to F L; and A C is to F L, as C B is to L I; and C B is to L I, as B A is to F I.

Fig. 7, 8. *Demonft.* If the Angle F be laid upon its Equal A, the Sides F I, F L, will fall upon the Sides A B, A C. And becaufe the external Angle A I L is by the Hypothefis equal to the internal B (*b*), therefore (*c*) I L, B C, are parallel. Therefore B I is to I A (*d*) as C L to L A. Therefore by compounding, B A is to I F, as C A to L F. And if the Angle L be laid upon the Angle C, it will be fhew'd in the fame manner, that A C is to F L, as B C is to I L; and if the Angle I be laid upon the Angle B, it will be fhew'd in the fame manner, that B C is to I L as A B to F I. The Propofition therefore is prov'd.

(b) *Fig.* 8. *alone.*
(c) *Per* 29. *l.* 1.
(d) *Per* 2. *l.* 6.

Corollaries.

Fig. 8. 1. IF in a Triangle a Line L I be drawn parallel to one Side B C, the Triangle L F I will be like to the Whole C B F; and confequently C F will be to L F, as B C to L I.

For becaufe L I, B C, are parallel, the external Angles F I L, F L I will (*per* 27. *l.* 1.) be equal to the internal ones B and C: But F is common to both Triangles There

Therefore they are equiangular. Therefore the Sides C F, L F opposite to the equal Angles B and F I L (*a*) are proportional to the Sides B C, L I, which are opposed to the common Angle F. (a) *By the foregoing.*

2. If in a Triangle a right Line B F, drawn from the opposite Angle B, doth cut the Parallels A C, L O, it cuts them proportionally. *Fig.* 9.

For by *Corollary* 1. A F is to L I, as F B is to I B; and F C also is to I O, as F B is to I B. Therefore A F is to L I, as F C to I O. Therefore by changing, A F is to F C, as L I to I O.

[3. From *Corollary* 1. " We learn to find the Heighth of a Tower, or any elevated Point, by only the Shadow of a Staff. Fix the Staff F L perpendicularly upon the Ground in that Place where the Ray of the Sun X B A, that terminates the Shadow of the Tower B Z may also pass through L. There will be in the Triangle A Z B, the Line F L, parallel to the Heighth of the Tower Z B. Whence as A F, the Distance of the Staff from the Point of the Shadow, is to F L, the Length of the Staff; so is A Z, the Distance of the Tower, from the Point of the Shadow. to Z B, the Heighth of the Tower. And because the three first Terms are easily had by measuring, the fourth also, *i. e.* the Heighth of the Tower is had also. *Q. E. I.* *Fig.* 51.

4. " From this also incomparably useful Proposition, we may deduce that famous Theorem of *Ptolomy*; to wit, that in every Quadrilateral inscrib'd in a Circle, the Rectangle of the Diagonals A C×B D is equal to the two Rectangles of the opposite Sides, A B×C D and A D×B C. For let the Angle B A E be made equal to the Angle C A D. Because the Angles B A E, C A D, are equal by Construction, the Angles A B E, A C D, standing upon the same Arch A D, are * equal; therefore the Triangles B A E, C A D, are alike. And A C : C D : : A B : B E; and consequently † the Rectangle of the Extremes A C×B E is equal to the Rectangle of the Means C D×A B. In like manner, because the Angle E A D is equal to the Angle B A C by Construction, and the Angles A D E, A C B, as standing upon the same Arch A B, are equal: The Triangles A D E, A C B, will be like; and A D : D E : : A C : C B. And therefore the Rectangle of the Extremes A D×C B is *Fig.* 52.

* *Per* 21. *l.* 3.
† *Per* 16. *l.* 6.

" equal to the Rectangle of the Means D E×AC. But the " Rectangles A C×B E, and A C×D E, are equal (*a*) to the " Rectangle A C×B D. Therefore the Rectangles AB×DC, " and A D×B C, which are made by the opposite Sides, " are equal to the Rectangle A C×B D, which is made by " the Diagonals. *Q. E. D.*

(a) *Per* 1. *l.* 1.

PROP. V. Theorem.

Fig. 10.

IF two Triangles have all their Sides proportional each to each, they shall also be mutually Equiangular.

That is, if A B be to R F, as A C to R Q; and as A C is to R Q, so is C B to Q F; and as C B is to Q F, so is A B to R F; I say, that the Angles opposite to the Antecedents, are equal to the Angles opposite to the Consequents; to wit, C to I, and B to F, and A to O.

Ang.	Antec.	Conseq.	Ang.
C	A B	R F	I
B	A C	R Q	F
A	C B	Q F	O.

Make X and Z equal to A and C; and let the Sides meet in N. The Angles B and N will (*a*) be also equal. Because therefore the Triangles P, T, are Equiangular, A B (by the foregoing) will be to R N, as A C to R Q. But by the Hypothesis, A B is to R F, as A C to R Q. Therefore A B is to R F, as the same A B is to R N. Therefore R N, R F, are equal. In the like manner, I might shew that Q N and Q F, are equal. Therefore the Triangles T, S, are equilateral to each other. Therefore the Angles, I, F, O, are equal (*per* 8. *l.* 1.) to the Angles Z, N, X, that is, by the Construction to the Angles, C, B, A. *Q. E. D.*

(a) *Per Corol.* 9. *p.* 32. *l.* 1.

PROP. VI. Problem.

Fig. 10.

IF two Triangles (P, S,) have an Angle (A,) equal to one Angle (O;) and the Sides (A B, AC, R F, R Q,) which contain the equal Angles proportional; the Triangles will be like.

Let

Let X and Z be made equal to the Angles A C, and the Sides meet together in N. Therefore the Angles B and N will † be alfo equal. Then it may be fhew'd, as in the foregoing, that R F, R N, are equal. But R Q is common to both Triangles S, T. The Angles alfo O and X are equal, becaufe they are equal to the fame A, the one X by the Conftruction, and O by the Hypothefis. Therefore (*c*) I and F are likewife equal to Z and N. Therefore the Triangle S is Equiangular to the Triangle T; that is, by the Conftruction, to the Triangle P. Therefore S, P are like (*per* 4. *l.* 6. *Q. E. D.*

(†) *Per Corol.* 9. *p.* 32. *l.* 1.

(c) *Per* 4. *l.* 1.

PROP. VII.

Is fcarce of any Ufe.

PROP. VIII. Theorem.

In a Rectangle Triangle, the Perpendicular (B C) let down from the right Angle to the Bafe, cuts the Triangle into Parts like to the Whole, and betwixt themfelves. *Fig.* 11.

In the Triangles A B F and L, the Angle F is common, but the Angles A B F and X are, by the Hypothefis right ones, and confequently equal. Therefore the other Angles A and O are (*b*) alfo equal. Therefore * the Triangles A B F and L are like: In the fame manner the Triangles A B F and R may be fhew'd to be equal, and the Angle I equal to the Angle F. From which it is now manifeft, that R and L alfo are like, feeing the Angles I and F, O and A, U and X, are equal. *Q. E. D.*

(b) *Per Corol.* 9. *Prop.* 32. *l.* 1.

* *Per* 4. *l.* 6.

Corollaries.

FIRST, BC is a mean Proportional betwixt A C and C F.

For feeing there be in the Triangle R and L,

equal Angles, I F	} {	equal Angles, A, O
Sides oppofed, A C, C B	} {	Sides oppofed, C B, C F.

(a) *Per* 4. *l.* 6. It is manifeft (*a*) that A C : C B : C F $\div$

2\. B F is a mean Proportional betwixt A F, and C F. Likewife A B, a mean betwixt F A, and C A.

For in the Triangle A B F and L.

equal Angles, A B F, X	} {	equal Angles, A, O
Sides oppofed, A F, B F	} {	Sides oppofed, B F, C F.

(b) *By the fame.* Therefore A F (*b*) : B F : : B F : C F. Likewife becaufe in Triangle A B F and R there be

equal Angles, A B F, V	} {	equal Angles, F, I
Sides oppofed, A F, A B	} {	Sides oppofed, A B, A C

It will be again A F : A B : A C $\div$

Fig. 11. 3. " Hence we learn to meafure an inacceffible Line, " one Term whereof is acceffible. Let the inacceffible " Line be C F. Let there be rais'd from the Point C the " Perpendicular C B : And to any Point of this Perpendi- " cular, as B, let there be applied a Square, or any right " Angle A B F ; fo that in looking along the Line B F, " the Point F, and along the Side B A, the Point A may " be obferved Let the acceffible Line A C be meafured, " and from the following Analogy the inacceffible C F will " be made known. AC : C B : : C B : C F. Let the Square " then of the Line C B be divided by the Line A C, and " the Quotient (*c*) will give the fought Line C F. *Q. E. I.*

(c) *Per Corol.* 3. *p.*17. *l.* 6.

PROP. IX. Problem.

Fig. 12. *TO divide a given Line (A B) according to a given Proportion F I to I L.)*

Let the infinite Line A Z be drawn. From which take A Q, Q R, equal to F I, I L. From R draw R B. Parallel to this, draw Q C from Q. I fay, the Thing is done.

It is manifeft from *Propofition* 2. *L.* 6.

PROP.

PROP. X. Problem.

TO divide a given Line, as (A B) in like manner as another given one (A I) hath been divided (in F, C.) *Fig.* 13.

Let the right Line I B join the Extremities of the two Lines. Draw Parallels to this from the Points, F. C, which may meet the right Line, that is to be cut, A B in L and Q. I ſay the Thing is done.

This is manifeſt from the *Corollary* of *Propoſition* 2. *l.* 6.

[" Or thus, if the cut Line I A be greater than that *Fig.* 53.
" which is to be cut, B Q, let three Circles touching one
" another, be deſcrib'd with the Diameter I F, I C, I A;
" and let the Subtenſe B Q be fitted from the Point I to
" the Circumference of the greateſt Circle: the two leſſer
" Circles will cut the Line B Q in the Points L, P, in the
" Proportion * of the Sections of the Diameter I A. If * *Per Co-*
" the Line I A be cut into four Parts, four Circles are to *rol.*4.*p.*31.
" be drawn; if into five, then five Circles; and ſo infinite- *l.* 3.
" ly.]

Scholium.

FROM this Propoſition we learn to cut a right Line given *Fig.* 13. into any equal Parts whatſoever. Let an infinite right Line make any Angle with the right Line to be cut, A B; from which take, with a Pair of Compaſſes, ſo many equal Parts, A C, C F, F I, as you would divide A B into. Draw the right Line I B, and the Parallels to it, F L, C Q. I ſay the Thing is done.

We may do the ſame Thing otherwiſe, and more eaſily *Fig.* 14. after *Maurolycus*, in the manner following. Let A B be to be triſected or divided into three equal Parts. Draw the infinite Line I X parallel to A B, above or below it. From I X, if it be below A B, take with a Pair of Compaſſes three equal Parts, I Q, Q R, R S, which together may be greater than A B; but leſſer, if I X is above. Through I and A, as likewiſe through S and B, draw right Lines which may meet together in C. From to Q and R draw right Lines: Theſe will triſect the given Line A B. The Demonſtration appears from *Corollary* 2. *Prop.* 4.

Fig. 15. Again, with *Maurolycus*, we may otherwise obtain the same thing, to wit, thus: Let A B be to be quadrisected. Draw the infinite Line A X, and B Z also an infinite Line parallel to it. From these take with the Compasses equal Parts A L, L O, O Q, and B V, V S, S R, in each fewer Parts by one than are required in A B; then let there be drawn the right Lines, L R, O S, Q V. These will quadrisect the given A B.

For because by Construction, the Lines L O, R S, parallel and equal, are join'd by L R and O S, these also (*a*) will be parallel. In the like manner OS and Q V are parallel. Therefore seeing A Q is cut into three equal Parts, A I will also (*b*) be cut into so many equal Parts. Likewise B C will be cut into three equal Parts. Therefore the whole AB will be cut into four equal Parts.

(a) *Per* 33. *l.* 1.

(b) *Per Corol. p.* 2. *l.* 6.

These two Ways of Practice are easier than *Euclid*'s, because fewer Parallels are to be drawn.

PROP. XI. Problem.

Fig. 16. *To find a third Proportional to two right Lines given (AB, BC.)*

Draw the right Line A C. From B A produced, take AF, equal to B C. Through F, draw the infinite Line F X, parallel to AC, which Infinite, let B C produced meet in L. I say that A B is to B C, as B C to C L.

For A B : A F (*b*) : : B C : C L. But A F (*c*) is equal to B C. Therefore A B : B C : : C L; and so C L is the third Proportional sought.

(b) *Per* 2. *l.* 6.

(c) *By the Construction.*

Otherwise.

Fig. 17. LET A B and B C be set at a right Angle. Join A C. From C draw C X perpendicular to A C infinite; which C X, let A B produced, meet, in L. I say, A B : B C : : B C : B L. It is manifest from *Corollary* 1. *p.* 8.

Scholium.

Scholium.

A Given Proportion may not only be continued in three, but alſo in infinite Terms, and the whole Sum of the infinite proportional Terms be exhibited. *Gregory* of *Saint Vincent* hath very handſomly proſecuted this Matter, and the whole Buſineſs of Geometrical Progreſſion in the whole Second Book of his Work. We, for the ſake of the Studious, will here preſent ſuccinctly the Conſtruction and Demonſtration of the Thing propoſed.

Problem.

LET a Proportion of the greater Inequality be given, as A B to B C. It is required to continue this thro' infinite Terms, and to preſent the Sum of them all. *Fig.* 19.

Let the Perpendiculars A L, B O, be erected, and taken equal to the given Lines A B, B C, and through L, O, let a right Line be drawn, meeting with A B C produced in Z. I ſay, 1. If from C you erect the Perpendicular C Q; C Q ſhall be a third Proportional. Transfer Q C into C E, and from E erect E R; this ſhall be a fourth Proportional. Transfer E R into E F, and erect F S; this ſhall be a fifth Proportional: And ſo the Proportion of A B to B C, that is, of A L to B O, will be continued through the Terms A L, B O, C Q, E R, F S, *&c.* or A B, B C, C E, E F, &c. infinitely, becauſe every Term (as F S) may be taken away from the remaining one F Z; for ſeeing L A (that is, A B) is leſs than A Z; F S alſo (*a*) muſt ever be leſs than F Z.

(a) *Per Corol.* 1. *Prop.* 4. *l.* 6.

I ſay, 2. A Z is equal to the whole Sum of the infinite Proportionals.

Part I. [" It being ſuppoſed as before, A Z : B Z : :
" A B : B C; it will be by alternating A Z : A B : : B Z :
" B C. And by dividing, A Z—A B : A B : : B Z—B C :
" B C; that is, B Z : A B : : C Z : B C. Therefore by
" inverting A B : B Z : : B C : C Z. And by compounding
" A B+B Z : B Z : : B C+C Z : C Z; that is, A Z : B Z
: : B Z

" : : B Z : C Z."] But as A Z is to B Z, so is L A to O B; and as B Z is to C Z, so is O B to Q C. Therefore also L A is to OB, as O B is to Q C. In the same manner I might shew that O B is to Q C, as Q C to R E; and so forwards infinitely.

Part II. The whole Sum of the infinite Terms is neither less than A Z, nor greater; therefore it is equal. It is not greater. because seeing we have shew'd above, that Q C is lesser than C Z, and R E than E Z, and S F than F Z, and so on infinitely, all the Terms Q C, R E, S F &c. may be infinitely set one by another in the right Line A Z; so that the Point Z shall never be reach'd. Again the said Sum will not be less, because I have above shew'd A Z, B Z, C Z, to be continually proportional; and in the same manner, the same thing is shew'd of the rest, E Z F Z, &c. Seeing therefore by transferring the Proportionals, Q C, E R, F S, &c. into C E, E F, F I, the Remainders E Z, F Z, I Z, &c. are always continually proportional, as we have already shewed; we shall at the last come unto a Remainder less than any given one; and therefore the Sum of the Proportionals shall exceed every Quantity that is less than A Z; from whence itself cannot be less than A Z. Seeing therefore it is neither greater nor less than A Z, it shall be equal to it. *Q. E. D.*

Theorem.

THE Difference of the first Terms, the first Term, and the whole Sum of the infinite Proportionals, are continually proportional.

Fig. 19. In the upper Figure let O X be drawn parallel to A Z. Therefore L X shall be the Difference of the first Term A L or A B, and of the second B O, or B C. Because X O is parallel to A Z; L X shall be to X O, as (*a*) L A is to A Z. But X O is A B, and L A likewise is A B. Therefore the Difference L X is to the first Term A B, as A B the first Term is to A Z, the whole Sum. *Q. E. D.*

(a) *Per Corol.* 1. *Prop.* 4. *l* 6.

Fig. 20. The same thing may be demonstrated universally and very briefly in every kind of Quantity; thus, Let there be any continual Proportionals whatsoever (as well Numbers, as other Quantities) A Z, B Z, C Z, &c. and let them all be transfer'd upon the first A Z. Therefore A B, BC, C E, E I

E F, F I, &c. will be the Differences of the Proportionals; which, together with the laſt Quantity I Z, are equal to the firſt A Z. Now, becauſe if Proportionals be continued infinitely, the laſt Quantity vaniſheth away, it is manifeſt that the Differences of the infinite Proportionals are equal to the firſt A Z. Then, becauſe A Z is to B Z, as B Z is to C Z, and ſo on. By dividing, A B will be to B Z, as B C to C Z; and by converting, as A B, the firſt Difference. is to A Z, the firſt Quantity; ſo B C the ſecond Difference, is to B Z, the ſecond Quantity, and ſo forwards. Therefore as A B, the firſt Difference, is to A Z the firſt Quantity. So all the Differences (that is, as I have already ſhewed, the firſt Quantity A Z) are to all the Quantities, that is, to the whole Sum of the infinite Quantities. *Q. E. D.*

PROP. XII. Problem.

THree right Lines being given (AB, BC, AF) to find a fourth Proportional.

Let the two right Lines be diſpoſed, as the Figure ſhews, and draw the right Line B F, to which let the infinite right Line C Z be made parallel. Let A F produced to L, meet C Z. *Fig.* 21.

I ſay, A B is to B C, as A F to F L. as is manifeſt from *Propoſition* 2. of this Book. Therefore F L is the fourth Proportional ſought.

Scholium.

OUR Countryman *Bettin*, in his Treaſury of Mathematical Philoſophy, doth handſomly from 35. *l.* 3. and 14 of this, which depends not upon the preſent Propoſition, find out a fourth Proportional, three being given, and a third, two being given, after this manner.

If three right Lines be given, let the ſecond C B, and the third B D, be join'd right to one another, ſo as to make one right Line, and let the firſt B A touch them in the Point B in what Angle you will. Through the Points C, A, D, deſcribe *Fig.* 22.

(a) *Per* 5. *l.* 4. describe a Circle (*a*), which let A B, the first Line, meet in the Point Z. B Z is a fourth Proportional.

(b) *Per* 35. *l.* 3. For seeing the Rectangles A B Z, C B D are (*b*) equal, A B will be to B C, as B D to B Z, by the 14th of this Book, which, as was said, depends not upon this.

Fig. 23. If there be given two right Lines, A B, B C; let B D, equal to B C, be join'd to B C, so as to make one strait Line. Then let the first A B touch B C in B in any Angle. Then the rest is as before, and B Z will be the third Proportional sought.

The Demonstration is the same; for seeing the Rectangles A B Z, C B D, are equal, A B will be to B C, as B D (that is, B C) is to B Z.

PROP. XIII. Problem.

Fig. 24. *TWO right Lines given (AC, CB) to find a mean Proportional.*

Let the whole compound Line A B be bisected in O, and from the Center O a Circle be described through A and B; from C erect a Perpendicular C F, meeting the Circumference in F.

I say, A C is to C F, as C F is to C B.

(c) *Per* 31. *l.* 3. For let A F, B F be drawn; the Triangle (*c*) A F B is right-angled, and from the right Angle there is drawn the Perpendicular F C to the Base. Therefore A C is to C F, as (*d*) C F is to C B.

(d) *Per Corol.* 1. *p.* 8. *l.* 6.

Corollary.

HEnce it is manifest, that if from any Point of the Circumference (as F) there be drawn a Perpendicular (F C) to the Diameter, this Perpendicular is a mean Proportional betwixt the Segments of the Diameter (A C, C B)

Scholium.

Scholium.

THIS Place requires, that we ſhould ſay ſomething briefly concerning the finding out of the two mean roportionals betwixt the two given Lines. All the Geoietricians of *Greece*, at *Plato*'s Suggeſtion, ſet themſelves vith all their Might to the Solution of this Problem. Divers ioſt ſubtil Ways of Practice are recited by *Eutocius* in his ommentary on *Archimedes*; as thoſe of *Plato*, *Architas* the *arentine*, *Menæchmus*, *Eratoſthenes*. *Philo Byzantius*, *Hero*, *pollonius* of *Perga*, *Nicomedes*, *Diocles*, *Sporus*, *Pappus*; to hom the later Times have added *Verner*, *Gregory* of Saint *incent*, *Renatus*, *Carteſius*. Out of all theſe we ſhall ſelect hree more eaſy than the reſt.

Plato's *Method.*

T is requir'd to find out two Means betwixt the given Lines A B, B C. *Fig.* 29.

Let A B, B C be ſet in a right Angle, and be produced nfinitely towards X and Z. Then let two Squares (ſo our *laudius Richards* hath it; for *Plato* himſelf made uſe of ne Square only, but which had inſerted into its Side * D E, Rule moveable along D E, let two Squares, I ſay, be aken, and the Angle D of one Square be applied to the right ine BX, in ſuch certain wiſe, that one Side may alſo paſs hrough A; and to the Point E, in which the other Side uts the right Line B Z, let a ſecond Square be applied, hich will paſs through C. I ſay, that B D, B E, are two Means betwixt the given Lines A B, B C; that is, as A B s to B D, ſo is B D to B E, and B E to B C.

* *See Fig.* 26.

The Demonſtration is manifeſt from *Corollary* 1. *Prop.* 8. . 6. for A D E is a right angled Triangle, and from the ight Angle to the Baſe there falls the Perpendicular D B. herefore by the ſaid Corollary, as A B is to B D, ſo is D to B E; and for the ſame Cauſe, as B D to B E, ſo is E to B C. Therefore betwixt the given right Lines A B, C, there are found two mean Proportionals B D, B E. Which was the Thing to be done. This Manner of ſolving he Problem is the eaſieſt of all to be underſtood.

The

The Method of Philo *the* Byzantine.

Fig. 27. LET the two given right Lines A B, B C, be ſet together at a right Angle; then let the Rectangle A B C D be perfected, and let D A, D C be produced infinitely, and let the Diameters B D, A C be drawn, cutting each other in E. From the Center E, through B, let a Circle be drawn, which, becauſe A B C is a right Angle (*a*) will paſs through A and C. Then let a Rule be applied to the Point B, ſo that the intercepted right Lines B G, O, F, may be equal. I ſay, that A F, G C, are two mean Proportionals betwixt the given A B, B C; that is, as A B is to A F, ſo is A F to G C, and G C to C B.

(a) *Per* 31. *l.* 3.

Demonſt. Becauſe G B, O F (*b*) are equal, O G, B F, will be alſo equal. Therefore the Rectangles O G B, B F O, that is, (*c*) the Rectangles D G C, D F A, are equal. Therefore as G D is to D F, ſo (*d*) reciprocally A F is to G C, but G D is to D F (*e*) as B A to A F. Therefore as B A is to A F, ſo A F is to G C. Again, becauſe I have already ſhew'd that A F is to G C, as B A is to A F; and ſince B A is to A F, as G D is to D F; that is, G C is to C B, A F will alſo be to G C as G C is to C B. Therefore all four, B A, A F, G C, C B, are continually proportional, and therefore betwixt the given Lines A B, B C, two Means have been found. *Q. E. I.*

(b) *By the Conſtruction.*
(c) *Per Corol* 1. *p.* 36. *l.* 3.
(d) *Per* 14. *l.* 6.
(e) *Per Corol.* 1. *p.* 4. *l.* 6.

Theſe two Methods of Solution, although they be ingenious and eaſy enough; yet becauſe a due Application of a Square and Rule is not made but by trying, they are not Geometrical.

The Method of Cartes.

Fig. 28. LET an Inſtrument of ſuch ſort be provided, that two Rules may be open'd and ſhut about Y. Let there be inſerted into theſe divers Squares connected together betwixt themſelves in the Points B, C, D, E, F, G. in ſuch ſort that in the mean while that the Rules Y X and Y Z are open'd, the Square B C may impel the Square C D in the Rule Y Z, and the Square C D may impel the Square D E in the Rule Y X, and the Square D E may impel F E, and

E

E F impel or force forward F G, and so on: But so that while the Rules X Y and Y Z are shut, all the Points B, C, D, E, F, G, tend to fall upon one and the same Point A. By this Instrument not only two, but also four and six, yea, as many Means as you will, betwixt two given right Lines, may be found. Which thing can be obtain'd neither by the Sections of a Cone, nor by any Methods found out by the abovesaid Authors.

For two Means, three Squares are required; for four Means, five Squares, and so on.

Let the lesser of the given right Lines be transferr'd upon the Rule Y X, and let it be Y B; the greater upon the Rule Y Z, and let it be Y E. Let the first Square be applied to the Point B, and be fixed there, and let the Rules be open'd, until the Side of the third Square passeth through E. I say, that YC, Y D, are two Means betwixt the given Y B, Y E, that is that Y B is to Y C, as Y C is to Y D, and Y D to Y E.

The Demonstration appears out of *Corollary* 2. *p.* 8. *l.* 6. For from the Nature of the Instrument, in the Triangle Y C D, the Angle at C is a right one, and from it C B falls perpendicular upon the Base Y D. Therefore by the said Corollary, as Y B is to Y C, so is Y C to Y D. Again, because in the Triangle Y D E, the Angle at D is a right one, and from it there falls the Perpendicular D C upon the Base Y E, as Y C is to Y D, so is Y D to Y E. Therefore Y B, YC, Y D, Y E, are four continual Proportionals. Betwixt the given Line therefore Y B, Y E, there have been found two mean Proportionals, Y C, Y D. *Q. E. I.*

If betwixt the given ones Y B, Y G, there be required four Means, open the Rules, until the Side of the fifth Rule Y G, passeth through G. There will be Y C, Y D, Y E, Y F, four Means betwixt Y B, Y G. The Demonstration is manifest from the said Corollary.

This Way, although the Instrument is more operose than *Plato*'s, is in very Deed an excellent one; both because it doth nothing by bare Tryal, and because it extends itself unto four and six, and as many Means as you will.

The *Deliacal* Problem, to wit, the Duplication of the Cube, is performed by two Means, and all the Bodies whatsoever are encreased or diminished in a given Proportion (*a*) by the same; like as the same thing is performed in plain

(a) *See Schol. p.* 18. *l.* 12.

(a) *Cor.* 3. *p.* 20. *l.* 6. plain Figures (*a*) by one Mean. *Hippocrates* first open'd this way, which, as the Singular and only one, all Geometricians that have followed him, have embraced.

PROP. XIV. Theorem.

Fig. 29, 30. *EQual Parallelograms (X, Z) which have one Angle (C) equal to one (O;) have their Sides also, which are about the equal Angles, reciprocal; that is, (AC is to CB, as FO is to OL.)*

And if they have the Sides thus reciprocal, the Parallelograms are equal.

Part I. Let IL and SB, being produced, meet togethe in Q. The Parallelogram X is to the Parallelogram R, a AC is to CB (*b*); and Z is to R (*c*), as FO to OL But because, by the Hypothesis, X and Z are equal, X i to R as Z is to R. Therefore also AC is to CB, as F(is to OL. *Q. E. D.*

(b) *Per* 1. *l.* 6.
(c) *By the same.*
(d) *By the same.*

Part II. As AC is to CB, so X is to R (*d*): And a FO is to OL, so is Z to R. But already by the Hypo thesis, AC is to CB, as FO to OL. Therefore X is to R as Z is to R. Therefore X and Z are equal. *Q. E. D.*

[Corollary. " On this depends the Demonstration of tl
" inverse Rule of Proportion. For in it there is alwa
" some Rectangle given, as X; and one Side of anoth
" equal Rectangle, as CB; and the other Side is sough
" As therefore AC, the first Side of the given Rectang
" is to CB, the given Side of the other Rectangle;
" reciprocally, FC, the sought Side, is to CL, the seco
" Side of the given Rectangle. The Rectangle therefc
" CB×FC, is equal to the Rectangle AC×CL: And t
" latter Rectangle given being divided by the given Side
" the former CB, the Quotient will give the sought Si
" FC. *Q. E. I.*

PROP. XV. Theorem.

Fig. 31, 32. *EQual Triangles (ACL, FCB) which ha one Angle (C) equal to one (O) have also the Sis*

Sides about the equal Angles reciprocal (that is, AC is to CB, as FO to OL.)

And if they have their Sides thus reciprocal, the Triangles are equal.

Let the right Line L B be drawn; the reſt of the Demonſtration is the ſame as that of the foregoing.

Corollary.

AS well Parallelograms as Triangles, which have their Baſes and Altitudes reciprocal, are equal: And ſo converſly.

It is manifeſt from the two foregoing Propoſitions.

PROP. XVI. Theorem.

IF four right Lines (AB, FI; IL, BC) be proportional, (that is, if AB be to FI as IL is to BC) the Rectangle (X) under the Extremes (AB, BC) is equal to the Rectangle (Z) under the Means (FI, IL.) Fig. 33.

And if a Rectangle under the Extremes be equal to a Rectangle under the Means, thoſe four right Lines will be proportional.

Part I. In the Rectangles X and Z, about the right, and therefore equal Angles, B, I, by the Hypotheſis, A B is to F I, as reciprocally, I L to C B. Therefore X and Z (*a*) are equal. *Q. E. D.* (a) *Per* 14. *l.* 6.

Part II. Becauſe X and Z are now ſuppos'd equal; therefore (*b*) about the equal Angles B and I, A B is to F I, as reciprocally, I L to B C. *Q. E. D.* (b) *By the ſame.*

[Corollary (1.) " Hence it is eaſy to apply the given " Rectangle Z (*c*) to the given right Line A B; to wit, " by making A B : F I : : I L : B C. For B C is the Rect- " angle Z applied to the given right Line A B) (c) *Per* 12. *l.* 6.

Corollary (2.) " Upon this Propoſition depends the " Demonſtration of the direct Rule of Proportion. For in

"it there is always given some Rectangle, as C L: and "another like Rectangle is sought, one Side whereof is also "given. It will therefore be, as B C, the first Side of "the Rectangle given, is to E O, the Side of the Rectangle sought; so directly C E, the second Side of the "Rectangle given, is to O A, the other sought Side. "Therefore the Rectangle C E×O E is equal to the Rectangle B C×O A. And the Rectangle C E×E O being divided by B C the Quotient, will give A O, the other "Side which was sought. *Q. E. I.*

PROP. XVII. Theorem.

Fig. 34. *IF the three right Lines (A B, F L, B C) be proportional, the Rectangle under the Extremes (A B, B C) shall be equal to the Square of the Mean (F L.)*

And if the Rectangle under the Extremes be equal to the Square of the Mean, those three right Lines are proportional.

Part I. Let O be taken equal to the Mean F L. Because therefore by the Hypothesis A B is to F L, as F L to B C, and O is equal to FL; A B will also be to F L, as O is to B C. Therefore (*a*) the Rectangle under the Extremes AB, B C, is equal to the Rectangle under the Means F L and O, that is, is equal to the Square of F L.

(a) *By the foregoing.*

Part II. This is demonstrated in like manner from the second Part of the foregoing.

Corollary.

Fig. 24. FROM this, taken together with the 13th, it is manifest that if in a Circle, F C be perpendicular to the Diameter, the Rectangle A C B is equal to the Square of F C.

(2.) If A×B be equal to the Square of C; then A : C : : C : B.

(3.) If A : C : : C : B; and C q be divided by A, the Quotient (*b*) will be B.

(b) *Per Corol.* 2. *p.* 16. *l.* 6.

PROP

PROP. XVIII. Problem.

UPON a given right Line (R S) to describe a Polygon like, and in like manner posited to a given one (B Q.) Fig. 35.

Resolve the given Polygon B Q into Triangles. Upon the given right Line R S, make the Angles (*a*) R, O, equal to the Angles B A. The Sides then will meet together in X. Upon X S make the Angles V, I, equal to the Angles T, C. The Sides then will meet together in Z. I say, the Thing is done.

(a) *Per* 23. *l.* 1.

For because the Angles R O, are equal to the Angles BA, the Angles E, K, must also be equal (*per* Corol. 9. *p.* 32. *l.* 1.) and because also by the Construction, V is equal to T, the whole E V must be equal to the whole K T. In like manner because O, I, are equal to A, C, respectively, the whole Angles O I, A C must be equal. And because V and I also are equal to T and C by the Construction, Z and Q likewise must be equal (*per* Cor. 9. *p.* 32. *l.* 1) to T and C. Therefore the Polygons R Z, B Q, are mutually Equiangular. It remains, that we shew that their Sides also are proportional: R S is to B F, * as S X to F L; and again, S X is to F L (*b*), as S Z to F Q. Therefore *ex æquo* R S is to S Z, as B F to F Q, &c.

* *Per* 4. *l.* 6.

(b) *By the same.*

Corollary. " Hence is derived the Method of making
" Maps or Charts, whether Geographical, or Chorogra-
" phical, or those which Surveyors of Land make ; and of
" framing Ichnographical Delineations of Fields, Buildings,
" Countries : for they are nothing else but the Reduction
" of great Figures unto like Figures which are of a small
" Compass, which is performed by the Means of this
" Proposition.

PROP. XIX. Theorem.

THE Proportion of like Triangles (X, Z) is duplicate of the Proportion of their Sides (A C, F I) which are subtended to the equal Angles. Fig 36, 37.

* *Per* 11. *l.* 6. That is, if it be made * as A C is to F I, ſo is F I to a third, A Q; the Triangle X is to the Triangle Z, as A C, the firſt, to the third Proportional, A Q. See *Definition* 10. 5.

Becauſe the Triangles X, Z, are like, B A will be to L I (c) *Per* 4. *l.* 6. (*c*) as A C is to I F. But by the Conſtruction, as A C is to I F, ſo is I F to A Q. Therefore alſo B A is to L I, (d) *Per* 15. *l.* 6. (*d*) as I F to A Q. Therefore in the Triangles Q B A and Z, the Sides about the Angles A, I, (which, by the Definition of like Triangles, are equal) are reciprocal. Therefore Q B A and Z are equal (*e*). (e) *Per* 1. *l.* 6. But the Triangle X is to Q B A, as the Baſe A C to the Baſe A Q (*f*). (f) *Per* 1. *l.* 6. Therefore X is to Z, as A C to A Q. *Q. E. D.*

Corollary. " Hence is their Error to be corrected, who " think that like Figures are in the ſame Proportion to one " another, that their Sides are. For if of two, not only " like Triangles, but alſo Squares, Pentagons, Hexagons, " *&c.* yea, and Circles alſo, the Sides or Diameters be " betwixt themſelves, as 2 to 1, the Figures or Areas " themſelves are as 4 to 1. If the Sides be betwixt them" ſelves, as 3 to 1, the Figures themſelves or Areas, are " as 9 to 1; to wit, in a duplicate Proportion of thoſe " Sides.

PROP. XX. Theorem.

Fig. 38. *LIKE Polygons (ABCDE, FGHIK) are divided,* (1) *Into like Triangles (P, S, and Q, T, and R, V) in Number equal.* (2.) *And proportional to the Wholes. And* (3.) *The Proportion of the Polygons is duplicate to that of the Sides, (AB, FG) which are betwixt the equal Angles (B, G, and BAE, GFK.)*

Part I. Becauſe the Polygons are alike, they are mutually (*per Def.* 1. *l* 6.) Equiangular, and their Angles equal B A E to G F K, and B to G, and B C D to G H I, and C D E to H I K, and E to K. Becauſe therefore A B is to B C (*a*) as F G to G H, and the Angles B and G are equal, (a) *By the ſame.* the Triangles P, S, (*b*) are like. (b) *Per* 6. *l.* 6. In like manner it will be demonſtrated, that R and V are like. Then, becauſe the Wholes, B C D, G H I, and the ſubducted ones, B C A G H F

G H F, are equal, the remaining ones alſo, A C D, F H I, are equal. In the ſame manner I might ſhew that A D C, F I H are equal. Therefore (*per* Corol. 9. *p.* 32. *l.* 1.) the third C A D is equal to the third H F I. Where alſo (*e*) the Triangles Q and T are alike. The firſt Part thereof is manifeſt.

(e) *Per* 4. *l.* 6.

Part II. Becauſe P and S are alike, the Proportion of P to S is duplicate to that of (*f*) C A to H F. But for the ſame Cauſe alſo the Proportion of Q to T is duplicate to the Proportion of C A to H F. Therefore P is to S as Q to T. In the ſame manner, I will ſhew that as Q is to T, ſo R is to V. Therefore, as one Antecedent, P, is to one Conſequent, S, ſo all the Antecedents, P, Q, R, taken together, are to all the Conſequents, S, T, V, taken together; that is, ſo is Polygon to Polygon. Which was the other Part

(f) *By the foregoing.*

Part III. The Proportion of P to S is duplicate (*h*) to that of A B to F G. But the Proportion of Polygon to Polygon is the ſame with the Proportion of P to S, as I have already ſhew'd. Therefore alſo the Proportion of Polygon to Polygon, is duplicate to the Proportion of A B to G F. Which was the third Part.

(h) *By the foregoing.*

Corollaries.

1. ALL ordinate or regular Figures, as Squares, Equilateral Triangles, Pentagons, *&c.* are betwixt themſelves in the duplicate Proportion of the Sides. For all regular Figures are like, as is manifeſt from *Definition* 1. 6.

2. If in any like Figures whatſoever, the Sides A B, F G, which are placed betwixt equal Angles, be known, the Proportion of the Figures is alſo known. As for Example; let A B be of two Feet, and F G of ſix Feet; and as 2 is to 6, ſo let 6 be to ſome other Number; to wit, 18. The leſſer Figure is to the greater, as 2 is to 18, or as 1 is to 9. Now a third proportional Number is found, if (*per* Corol. *. p.* 17. *l.* 6.) the ſecond of the given ones be multiplied by itſelf, and the Product divided by the firſt. *Fig.* 38.

3. From the ſame Propoſition is drawn the excellent Method of encreaſing or diminiſhing any Rectilineal Figure in a given Proportion. As if I would make a Pentagon, whoſe Side is A B, five fold of another. Find a mean Proportional, B X, (*i*) betwixt the Terms of the Proportion given, *Fig.* 39.

(i) *Per* 13. *l.* 6.

(a) *Per* 18. *l* 6. given, AB, B C; upon this Frame, (*a*) a Pentagon like to the given one. This ſhall be quintuple of the given one.

For by the 20th, the Pentagon A B is to B X, which is like to it, as A B, the firſt, is to B C, the third Proportional.

Moreover, ſeeing the Proportion of Circles alſo is duplicate to the Proportion of their Diameters, as will be ſhew'd, *p.* 2. *l.* 12. This Practice belongs likewiſe to Circles.

Fig 41. [Scholium. " Seeing the Proportion of the Squares " E, K, is duplicate of the Proportion of their Sides O R, " S V; from thence the duplicate Proportion of the Sides " O R, S V, is wont commonly to be expreſs'd by the " Proportion of O R q to S V q.]

PROP. XXI. Theorem.

Fig. 40. *Figures A, (B) which are like to the ſame (C) are alſo like betwixt themſelves.*

This is manifeſt from *Definition* 1. *Lib.* VI. and from *Axiom* 1. *Lib.* I.

PROP. XXII. Theorem.

Fig. 40, 41. *IF four or more right Lines (FI, LQ, and OR, SV,) be proportional; like Figures, and in like Sort deſcribed by them (A B and E K) muſt alſo be proportional.*

And Converſly.

The Demonſtration of the firſt Part is manifeſt. For becauſe the Proportions of A to B and E to K, are duplicate to the Proportions of F I to LQ, and O R to S V, which are, by the Hypotheſis, equal; themſelves alſo muſt be equal.

The ſecond Part is manifeſt alſo.

Fig. 24. [Corollary. " If the right Line AB be cut in any manner in C; the Rectangle contain'd under the Parts A C

C B

" CB, is a mean Proportional betwixt their Squares. " Likewise the Rectangle contain'd under the Whole A B, " and one Part, A C or CB, is a mean Proportional be- " twixt the Square of the Whole, AB, and the Square of " the said Part, AC or CB, For (*per* Cor. 1. *p.* 8. *l.* 6.) " it is manifest, that * A C : F C : : C F : C B. There- " fore A C Square : C F Square : : C F Square : C B Square. " That is, * A C Square : Rectangle A C B : : Rectangle " A C B : C B Square. *Q. E. D.*

Corol. 1. *p.* 8. *l.* 6.

* *Per* 17. *l.* 6.

" Moreover, (*per* Cor. 2. *p.* 8. *l.* 6.) B A : A F : : A F : " AC. Therefore B A q : A F q : : A F q : A C q. That " is, † B A q : B A C Rectangle : : B A C Rectangle : " A C q. In the same manner A B q : A B C : : A B C : " B C q. *Q. E. D.*

† *Per* 17. *l.* 6.

PROP. XXIII. Theorem.

EQuiangled Parallelograms (X, Z) have betwixt themselves a Proportion that is compounded of the Proportions of their Sides (A C CB, and L C to C F.) Fig. 42.

That is, if you make C B to be to O, as L C to C F, X is to Z, as A C is to O.

Let I L, S B, meet together in Q. The Parallelogram X (*a*) is to the Parallelogram R, as A C is to C B ; and R is (*b*) to Z, as L C is to C F ; that is, as C B is to O. Therefore *ex æquo* X is to Z, as A C is to O. *Q. E. D.*

(a) *Per* 1. *l.* 6.

(b) *By the same.*

Corollaries.

FROM hence, and from 34 *l.* 1. it is manifest.

1. That Triangles which have one Angle (at C) equal, have that Proportion betwixt themselves, which is compounded of the Proportions of the right Lines A C to C B, and L C to C F. Which Lines contain the equal Angle. *Fig.* 42.

2. That Rectangles, and consequently all Parallelograms whatsoever, have betwixt themselves the Proportion which is compounded of the Proportions of the Base to the Base, and the Height to the Height. And in the same manner we reason about Triangles.

Fig 42. 3. Hence the Proportion of Triangles and Parallelograms may be readily learned. Let X and Z be the Parallelograms, and their Bafes AC, CB, and CL, CF, be their Heights. Let it be made (c) as the Altitude CL, is to the Altitude CF, fo is one of the Bafes CB, to O. The Parallelogram X is to the Parallelogram Z, as AC to O.

(c) *Per* 12. *l.* 6.

PROP. XXIV. Theorem.

Fig. 43. *IN every Parallelogram (as S, F) the Parallelograms which are about the Diameter (AB) to wit, (CL, OI) are both like to the whole Parallelogram, and to each other.*

By 27. 1. the Angles, C, S, and L, F, are equal. By the fame, E is equal to I, that is, by the fame, equal to A it felf; but B is common both to the Whole, S, F, and the Part, CL. Therefore the whole, S, F, and the Part, CL, are Equiangular. It remains to be fhew'd, that they have the Sides oppofite to the equal Angles proportional.

Becaufe in the Triangle BCE, BSA, CE is parallel to SA, BC (by *Corol.* 1. *p.* 4. *l.* 6.) will be to CE, as BS to SA: And CE will be to EB (by the fame *Corollary*) as SA to AB. But becaufe in the Triangles ELB, AFB alfo, EL is parallel to AF, EB (by the fame *Corollary*) will be to EL, as AB to AF. Therefore *ex æquo* CE is to EL, as SA to AF. Therefore (by *Definition* 1. *L.* VI.) CL, and the Whole, CF, are like. In the fame manner, I might fhew OI to be like to the Whole. S, F. Therefore (*per* 21. *l.* 6.) CL and OI are alfo like betwixt themfelves. *Q. E. D.*

PROP. XXV. Problem.

Fig. 46. *TO change a given Polygon (A) into another like to a given one (B.)*

Or to make a Polygon equal to a given one (A) and like to another given one (B.)

Upon CF, the Side of the Polygon B, a like one to which is required, (by 45 *l.* 1.) make a Rectangle Q equal to B. Then upon FI (by the fame *Propofition*) make a

Rect-

Rectangle R equal to A. It is manifest, that C F and F I do make one right Line. Betwixt C F and F I find a mean Proportional F L (*a*). Upon this, (*p.* 18. *l. 6.*) make a Polygon like to the given one B, this must also be equal to the given one A. (a) *Per* 13. *l.* 6.

For seeing by the Construction, C F, F L, F I, are three Proportionals, the Polygon B is to the Polygon like to it, which is made upon F L, as C F is to F I (*per* 20. *l.* 6. and *Definition* 10. *l.* 5. that is, (*per* 1. *l.* 6.) as Q is to R. Therefore also by changing, as the Polygon B is to Q, so is the Polygon F L to R. But by the Construction, the Polygon B is equal to Q. Therefore also the Polygon upon F L, which is like to B, is equal to R; that is, by the Construction, to the given A. That therefore is done which was required.

PROP. XXVI. Theorem.

LIKE Parallelograms (B D, F N) having a common Angle (A) are about the same Diameter. Fig. 44.

Draw the right Lines A E, C E. If you deny that A E C is a common Diameter to the Parallelograms B D and F N; let another right Line A G C, which cuts F E in G, be the Diameter of B D, and draw the Parallel G H. The Parallelograms F H, B D, will be therefore about the common Diameter A G C, and consequently (by 24. *l.* 6. will be like. Therefore, (*per Definition* 1. *l.* 6.) will be like. Therefore, as B A to A D, so is F A to A H. But also, as B A to A D, so is F A to A N, seeing B D, F N, are like by the Hypothesis. Therefore F A is to A H, as the same F A is to A N. Which is absurd.

PROP. XXVII, XXVIII, XXIX.

THESE cause Trouble to, and perplex Beginners, and are scarce of any Use.

PROP.

PROP. XXX. Problem.

Fig. 45. *TO cut a given right Line (AB) ſo that the whole (AB) ſhall be to one Segment (AC) as the ſame Segment is to the Remainder (CB.)*

That is, as Geometricians ſpeak, to cut a Line in extreme and mean Proportion.

By 11 *L.* 2. ſo cut A B in C, that the Rectangle under A B, C B, may be equal to the Square of A C. I ſay the Thing is done.

For by the 17th of this Book, as A B is to A C, ſo is A C to C B.

The Force of this Section of a Line is admirable in the inſcribing and comparing regular Bodies.

PROP. XXXI. Theorem.

Fig. 47. *IF from the Sides of a Rectangular Triangle (ACB) like Figures whatever be deſcribed, that which is oppoſed to the right Angle, will be equal to the two others (L, R) taken together.*

Here *Propoſition* 47. *l.* 1. is made univerſal.

From the right Angle C, let the Perpendicular C O be let down. Becauſe (*per* Corollary 2. *P.* 8 *L.* 6.) A B, B C B O, are three Proportionals, F ſhall be to the Figure R which is like to it, as A B the firſt, to B O the third Proportional, (to wit, by 20 *L.* 6. and *Definition* 10. *L.* 5.) Again, becauſe (by the aforeſaid Corollary) B A, A C A O, are three Proportionals, the Figure F ſhall (by the aforeſaid *Propoſition* and *Definition*) be to L, which is like to it, as B A the firſt, to A O the third Proportional. Becauſe therefore F is to R, as A B is to B O; and the ſame F is to L, as A B to AO; F ſhall alſo be to R and L taken together, as A B is to B O, A O, taken together. But A B is equal to the two, B O, A O. Therefore alſo F ſhall be equal to the two, R and L. *Q. E, D.*

Corollary

Corollary.

FROM this Propofition we can eafily find one Rectilinear Figure, equal and like to any Number of Rectilinear Figures whatfoever, by the fame Method, whereby, *Prop.* 1. *Schol. p.* 47. *l.* 1. one Square is found equal to any Number of given Squares whatfoever. Only in the Demonftration, let 31. *l.* 6. be cited inftead of 47 *l.* 1.

Corollary (2.) " A Circle upon the Hypothenufe of a " Rectangle Triangle, is equal to two Circles defcribed " upon the Sides, for all Circles are like amongft them- " felves; and are to one another as the Squares of their " Diameters, by the Second of the Twelfth Book.

Corollary (3.) " From hence we may derive that Qua- " drature of Lunets (or little Moons) which *Hippocrates* " of *Chios* firft taught. *Fig.* 54.

" For let A B C be a Rectangle Triangle; and B A C a " Semi-circle to the Diameter B C: B N A a Semi-circle " defcrib'd on the Diameter A B; A M C a Semi-circle " defcrib'd upon the Diameter A C. Thus therefore the " Semi-circle B A C is equal to the Semi-circles B N A and " A M C together. If therefore you take away the two " Spaces B A, A C, common on both Sides, there will be " left the two Lunets B N A, A M C, bounded on both " Sides with circular Lines equal to the Rectilineal Tri- " angle B A C. And if the Line B A be equal to the Line " A C, and you let fall a Perpendicular unto the Hypothe- " nufe B C, the Triangle B A O will be equal to the Lunet " B N A, and the Triangle C O A equal to the Lunet C M A. *Q. E. I.*

PROP. XXXII.

THIS is hardly of any Ufe, and hath nothing remarkable in it.

PROP.

PROP. XXXIII. Theorem.

Fig. 48. *IN the ſame or equal Circles, the Angles, whether at the Centers (as ABC, FOD) or at the Circumference (as ARC, FSD) have that Proportion betwixt themſelves, which the Arches (AKC, FGD) on which they ſtand, have. Underſtand the ſame Thing of Sectors.*

As for the Angles at the Center, and the Sectors, it will be demonſtrated altogether in the ſame manner, in which, *Prop.* 1. of this Book, it was demonſtrated, that Triangles of the ſame Height are as their Baſes: Only where, *Prop.* 38. *l.* 1. is cited there, let *Prop.* 29. *l.* 3. be cited here.

And becauſe the Angles R and S, at the Circumference, are Halves of the Angles A B C, F O D, at the Center, that which hath been demonſtrated of theſe will be manifeſt alſo of thoſe:

Corollary.

Fig. 49. 1. THE Angle (B A C) at the Center, is to four right Angles, as the Arch B C on which it ſtands, is to the whole Circumference.

For as B A C is to the right Angle B A F, ſo by this, 33, the Arch B C is to the Quadrant B F. Therefore the Angle B A C is to four right Angles, as the Arch B C is to four Quadrants, that is, the whole Circumference.

The Arches I L, B C, of unequal Circles, which do ſubtend equal Angles, whether at the Center, as I A L and B A C, or at the Circumference, are like Arches.

For the Arch I L is (by *Corollary* 1.) to its Circumference, as the Angle I A L, that is, B A C is to four right Angles; and the Arch B C is to its Circumference (by the ſame *Corollary*) as the ſame Angle B A C is to four right ones. Therefore I L is to its Circumference, as B C is to its. Therefore (by *Defin.* 4. *l.* 6.) the Arches I L and B C are like.

3. The

3. The Semi-diameters (A B, A C) do take away from concentrical Circumferences like Arches, I L, B C. This is manifeſt from *Corollary* 2.

4. The Segments (B K C, I O L) which contain equal Angles (K, O) are like.

For by *Corollary* 2. the Arches B C, I L, and conſequently the Angles B K C, I O L, are like.

THE

Elements of EUCLID.

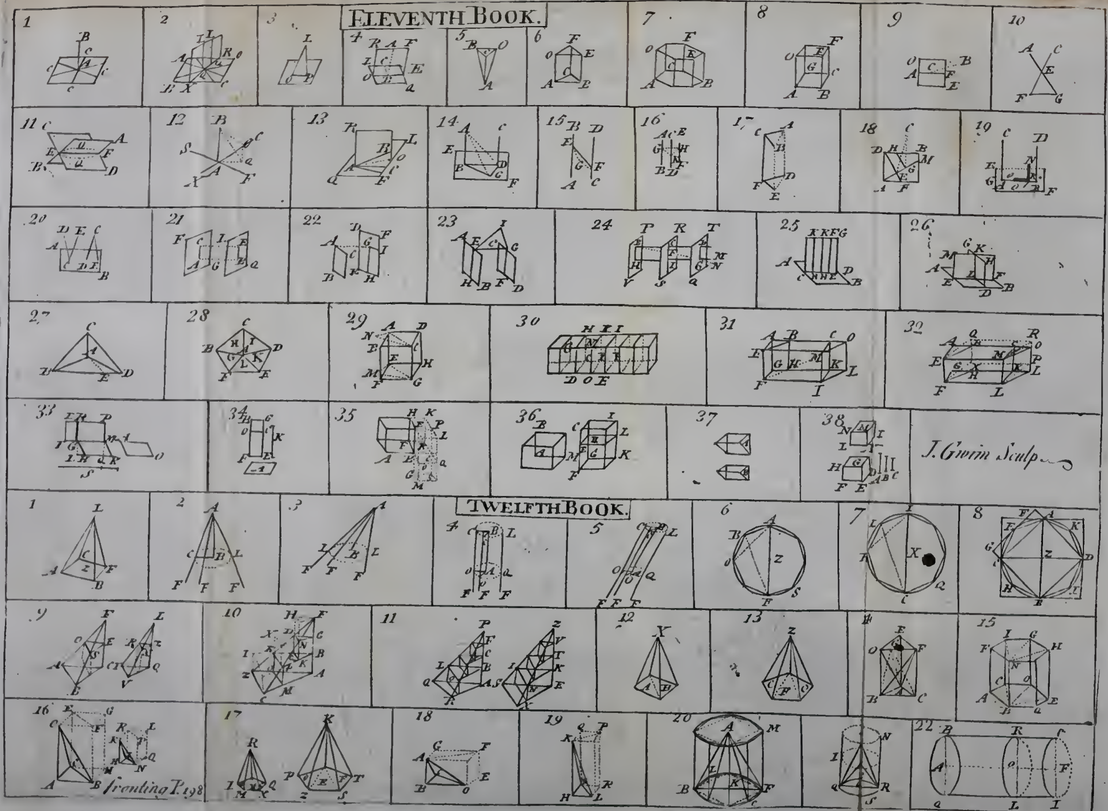
BOOK XI.

With Us the Seventh.

TO the ſix firſt Books *Euclid* ſubjoins the Elements of Numbers, comprehended in the three following, the Seventh, Eighth and Ninth, to which he alſo adjoins a Tenth, concerning incommenſurable Quantities. We paſs immediately from Planes to Solids; purpoſing to treat of Numbers ſeparately: Seeing it will, I ſuppoſe, be more commodious for Learners, if the Elements of Geometry be not interrupted, by treating of any other Matter, but be had altogether. Nevertheleſs, when we ſhall cite the Propoſitions of this and the following Book, we ſhall not call theſe Books the Seventh, and the Eighth, but the Eleventh and the Twelfth, leſt if we ſhould depart from the every where received Order of *Euclid*, the Citation of Propoſitions ſhould thereby be render'd more intricate.

This Book in a ſort contains two Parts: In the firſt, are laid the Foundation on which the whole Doctrine of the Solids, that is, of Bodies, depends. In the other, the Affections of Parallelopepids are propounded.

" This Eleventh Book of Elements ſets forth the firſt
" Principles of Solids. Nor can indeed the Properties of
" Bodies be known without it; and if we ſet upon almoſt
" any Part of the Mathematicks, without the Knowledge
" of Solids, we ſhall labour in vain, or be at leaſt at a great
Loſs.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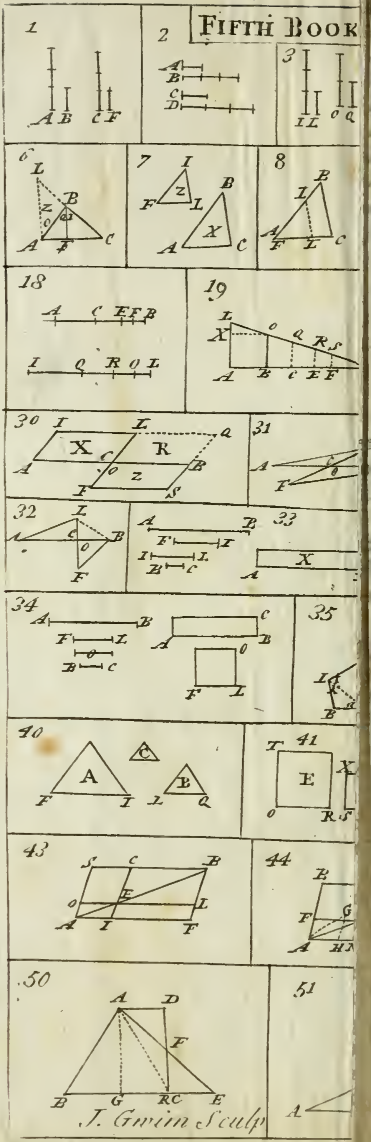
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Loss. For the Spherical Doctrine of *Theodosius*, Spherical Trigonometry also, a great Part of Practical Geometry, Statics and Geography, depends upon it; and what Things occur of any great Difficulty in the Art of Dialling, in the Conic Sections, Astronomy, Dioptrics or Optics, do all become more easy, the Principles of Solids being once understood: So that those who have delivered the Elements of Geometry, leaving out and setting aside this and the following Book, are to be reckon'd to have delivered the same very imperfectly.

DEFINITIONS

1. A Solid, or Body, is that which hath Length, Breadth and Thickness.

2. The Extreme of a Solid is a Surface. *Fig.* 1. *l.* 11.

3. The right Line [AB] is to the Plane [CC] right or perpendicular, when it makes right Angles [B A C, B A C] with all the right Lines [C A] in the Plane [CC] by which it is touch'd.

4. A Plane is right or perpendicular to a Plane, when all the right Lines [L Q] which are drawn in one of the Planes perpendicular to the common Section [X R] are right or perpendicular to the other Plane [A B C O.] *Fig.* 2.

5. If the right Line [O L] stands upon a Plane not at right Angles, and from its highest Point [L] there be drawn to the Plane the Perpendicular [L P] and [O P] be join'd; the Angle [L O P] is said to be the Inclination of the Line [O L] to the Plane. *Fig.* 3.

6. If the Plane [R E] doth not stand perpendicularly upon the Plane [L Q] the Inclination of one to the other is the acute Angle [A B C] which is contain'd by the right Lines [A B and B C] which are drawn in both Planes perpendicular to the common Section [O E.] *Fig.* 4.

7. A Plane is said to be alike inclin'd to a Plane, as is some other Plane to another, when the said Angles of their Inclinations are equal.

8. Parallel Planes, are those which being continued every way, are always distant from each other by equal Intervals.

9. Like solid Rectilinear Figures are those which are contain'd under like Planes, in Number equal.

10. A solid right-lin'd Angle, is that which is contain'd under plain Angles more than two [B A C, C A O, O A B] *Fig.* 5.

which

which are not in the ſame Plane, meeting together in on Point.

11. Equal ſolid Angles are thoſe, which being conceiv' to be put each within the other, do agree or perfectly co incide.

Like as a plain Angle is a mutual Inclination of Lines ſo a ſolid Angle is an Inclination of Surfaces. Concernin both therefore we muſt reaſon in the ſame manner.

Fig. 6, 7, 8. 12. A Priſm is a ſolid Figure, comprehended by Planes amongſt which two oppoſite ones [O F E, A C B] are pa rallel, equal and like.

Fig. 8. 13. A Parallelopepid is a Solid, contain'd under ſi Quadrilateral Planes, of which the Oppoſites are paralle

14. If ſix Planes, in which the Oppoſites are paralle be Squares, the Solid contain'd by them will be a Cube.

PROPOSITION I. Theorem.

Fig. 9. *ONE Part (AC) of a right Line cannot be i a Plane (OE;) and another Part (CB) ou of it.*

It is clear of it ſelf, from the Definition of a Plane an a right Line. See *Defin.* 4. and 7. *L.* 1.

PROP. II. Theorem.

Fig. 10. *EVERY Triangle is in one Plane: And tw right Lines cutting each other, are in th ſame Plane.*

For if a Plane be applied to one of its Sides, and to th Point of meeting the other two, it will be evident that th whole Triangle is in that Plane.

PROP. III. Theorem.

Fig. 11. *IF two Planes (AB, CD) cut each other (EF their common Section is a right Line.*

It is manifest from the Definition of a Plane.

But we may demonstrate it thus. If E F, the common Section, be not a right Line, let there be drawn in the Plane C D the right Line E O F, and in the Plane A B, the right Line E Q F. The two right Lines therefore, E O F, E Q F, will inclose a Space. Which is absurd.

PROP. IV. Theorem.

If a right Line (B A) be perpendicular to two right Lines (C A X, F A S) which cut each other, it will also be perpendicular to the Plane which is drawn through them. Fig. 12.

If you deny it, let another right Line, B Q, be perpendicular to the Plane of the right Lines A C, A F. Join A Q, and to this in the Plane F A C, draw the Perpendicular Q O. This being produced, will necessarily cut (as is gathered from *Schol. Prop.* 31. *l.* 1.) one of the right Lines C A X, F A S, or both, wheresoever the Point Q shall be. Therefore let it cut C A X in O, and let B O be join'd. Because therefore the Angle B A O is, by the Hypothesis, a right one;

The Square of B O shall be equal to B A Squ. + A O Squ. (*b.*) — (b) *Per* 47. *l.* 1.

But because B Q is suppos'd perpendicular to the Plane F A C, and consequently (by *Definition* 3. *l.* 11.) makes a right Angle B Q A with A Q;

B A Squ. is equal to B Q Squ. + A Q Squ. (*d.*) — (d) *Per* 47. *l.* 1.

And because the Angle A Q O is, by the Construction, a right one;

A O Squ. is equal to O Q Squ. + A Q Squ. (*e.*) — (e) *By the same.*

Therefore B O Squ. is equal to + B Q Squ.
+ O Q Squ.
+ A Q Squ. twice taken.

Therefore Square B O is greater than the Squares of B Q and O Q; and (as is clear from *Prop.* 47 *l.* 1. consequently B Q O is not a right Angle. Therefore B Q is not perpendicular to the Plane (by *Definition* 3. *l.* 11.) C A F. Therefore the Proposition is manifest.

Scholium.

FROM its being suppos'd that B Q is perpendicular to the Plane F A C; it is directly demonstrated that B Q is not perpendicular to that Plane; and consequently from the denial of the Assertion of the Theorem, the same Assertion is directly proved. This Demonstration, as to the Substance of it is *John Cierman*'s.

PROP. V. Theorem.

Fig. 13. *IF three right Lines (B A, C A, F A) be perpendicular to the same right Line (A R) at the same Point (A;) those three will be in one Plane.*

For, if it may be, let one of them B A be in another Plane (R O) which may cut L Q, the Plane of the other two, C A, F A, in the right Line A O. Because, by the Hypothesis, R A stands perpendicularly upon the two, C A, F A, it will be perpendicular to the Plane L Q by the foregoing.) Therefore R A makes a right Angle with A O (by *Definition* 3. *l.* 11.) But also, by the Hypothesis, R A B is a right Angle. Therefore the Angles R A B and R A O are equal. Which is absurd.

PROP. VI. Theorem.

Fig. 14. *RIGHT Lines (A B, C D) which are perpendicular to the same Plane (C F) are parallel.*

It might be taken for granted, as a Thing of itself known; but we may demonstrate it thus.

B D being join'd, make in the Plane F E the Line D G perpendicular to B D, and equal to B A; and let D A, G A, G B, be join'd. The right Lines B D, D G, are equal to B D (*a*) and B A; and the Angles B D G, (*b*) D B A are right ones. Therefore (*per* 4. *l.* 1.) A D, B G, are equal. Therefore the Triangles A B G, G D A, are Equilateral to each other, and consequently the Angles A B G, A DG, are equal. But A B G (by *Defin.* 3. *l.* 11.) is a right Angle. Wherefore A D G is also a right one. But B D G also, by the Construction, and C D G, by *Defin.* 3. are right Angles. Therefore G D is perpendicular to the three Lines C D, A D, B D. Therefore C D is (*c*) in one Plane with A D and B D. But A B also is in one Plane (*per* 2. *l.* 11.) with A D and B D. Therefore A B, C D are in one Plane. Therefore seeing the Angles A B D, C D B (by *Defin.* 3. *l.* 11.) are right ones, A B, C D, will (*per* 29. *l.* 1. and *Defin.* 36. *l.* 1.) be parallel Lines. *Q. E. D.*

(a) *By the Construction.*
(b) *Per Def.* 3. *l.* 11.
(c) *By the foregoing.*

PROP. VII. Theorem.

A Right Line (E F) cutting the right Lines (AB, CD) placed in the same Plane, is in one and the same Plane with them. Fig. 15.

It might be taken for granted. But he that will may thus demonstrate it.

Let another Plane cut the Plane of the right Lines A B, CD, in the Points EF. If now E F is not in the Plane of AB, C D, E F will not be the common Section. Let E G F therefore be so. Therefore (*per* 3. *l.* 11.) E G F is a right Line; the two right Lines therefore E F, E G F, inclose a Space. Which is absurd.

Corollary.

HENCE it follows, that if E F cut the Parallels A B, C D, it is in the same Plane with them. For (by *Definition* 36. *l.* 1.) any two Parallels are in the same Plane.

PROP. VIII. Theorem.

Fig. 14. *If of two Parallels (AB, CD) one (AB) be perpendicular to a Plane EF; the other also (CD) will be perpendicular to the same Plane.*

It might be taken for granted. If the Demonstration be requir'd, it is as follows.

" BD, AD, being drawn: in the Plane EF, make " GD perpendicular to BD. It will also (see the Demon- " stration of *Prop.* 6. *l.* 11.) be perpendicular to AD. " Therefore (*per* 4. *l.* 11) GD will be perpendicular to " the Plane ABD, that is, (by the foregoing Corollary) " to the Plane CBDA. Wherefore (*per Defin.* 3. *l.* 11.) " CDG is a right Angle. But the Angle CDB is also a " right one; forasmuch as with ABD, which (*per Defin.* " 3. *l.* 11.) is a right Angle, it maketh two right ones " (*per* 27. *l.* 1.) Therefore (*per* 4. *l.* 11.) CD is perpen- " dicular to the Plane GDB or EF. *Q. E. D.*

PROP. IX. Theorem.

Fig. 16. *Right Lines (AB, EF) which are parallel to the same right Line (CD) although they be not in the same Plane with it, are also parallel betwixt themselves.*

Although it might be taken for granted, yet we will demonstrate it thus.

In the Plane of the Parallels AB, CD, draw GK perpendicular to CD. Likewise in the Plane of the Parallels EF, CD, draw HK perpendicular to CD. Therefore
(a) *Per* 4. *l.* 11. (*a*) CK is perpendicular to the Plane GKH. Therefore, seeing AG, EH, be parallel to CK, the same AG, EH
(b) *Per* 8. *l.* 11. (*b*) will be perpendicular to the Plane GKH. Therefore AG, EH (*c*) are parallel. *Q. E. D.*
(c) *Per* 6. *l.* 11.

PROP.

PROP. X. Theorem.

IF two right Lines (A C, B C) be parallel to two right ones (D F, E F;) albeit they be not in the ſame Plane, they comprehend equal Angles (C and F.) Fig. 17.

Let C A, C B, be made equal to F D, E F, and let D E, A B, D A, F C, E B, be drawn. Seeing A C, F D, are parallel and equal, A D alſo and C F will (*a*) be parallel and equal. In like manner I might ſhew B E, C F, to be parallel and equal. Therefore A D, B E, are alſo parallel (*b*) and equal, *per Axiom* 1. Therefore, *per* 33. *l.* 1. A B, D E, are equal. Seeing therefore the Triangles B A C, E D F, are Equilateral to each other, the Angles C and F (*c*) are equal. *Q. E. D.*

(a) *Per* 33. *l.* 1.

(b) *By the foregoing.*

(c) *Per* 8. *l.* 1.

PROP. XI. Problem.

TO draw a Perpendicular to a given Plane (A B) from a Point given without it (C.) Fig. 18.

The Conſtruction. In the Plane A B, draw any right Line. as D F, unto which, from C, erect the Perpendicular C E. Then in the Plane A B, through E, draw A E M perpendicular to the ſame D F. Then to A M, from C, draw the Perpendicular C G. I ſay, that C G is perpendicular to the Plane A B.

Through G let H G be drawn parallel to D F. By the Conſtruction, D E is perpendicular to C E and E M. Therefore D E is perpendicular to the Plane C E M (*d*), as alſo is H G (*e*). Therefore, by *Defin.* 3. *l.* 11. C G is perpendicular to H G. But C G, by the Conſtruction, is alſo perpendicular to E M. Therefore (*f*) C G is perpendicular to the Plane A B. Which was the Thing propoſed.

(d) *Per* 4. *l.* 11.

(e) *Per* 8. *l.* 11.

(f) *Per* 4. *l.* 11.

Scholium. " In Practice thus. Let there be a Cord ' or Rule faſtned to the given Point A: And from the ' ſame, let there be deſcribed by the end of it B in the ' Plane given, the Circle B C F L. The Line A K, ' which connects the given Point and the Center of the ' Circle, will be perpendicular to the given Plane. Fig. 20. *l.* 12.

PROP.

PROP. XII. Problem.

Fig. 19. *FROM a given Point (A) in any Plane (EF) to erect a Line perpendicular to the ſame Plane.*

From any Point D, without the Plane E F, make D B (by the foregoing) perpendicular to the Plane E F. And B A being join'd, draw A C parallel to D B. I ſay the Thing is done. The Demonſtration is manifeſt from *Prop.* 8.

Corollary.

IN Practice, from the given Point, a Perpendicular is erected to the given Plane, if a Square O K N be applied to the given Point [and be turn'd round.]

PROP. XIII. Theorem.

Fig. 20. *LINES drawn from the ſame Point cannot be both perpendicular to the ſame Plane (AB.)*

For if they were, they would, (by *Prop.* 6.) be parallel Which cannot be.

PROP. XIV. Theorem.

Fig. 21. *IF the ſame right Line (AB) be perpendicular to two Planes (FG, LQ;) the Planes will be parallel.*

Let there be taken in either of the Planes, as F G, any Point C, from which let C E be drawn parallel to A B and meeting the Plane L Q in E. Then C E (*per* 8. *l.* 11.) will be perpendicular to both Planes, F G, L Q. Wherefore if A C, B E be join'd, the Angles A, B, (by *Def.* 3. *l.* 11.) will be right ones. Therefore (*per* 29. *l.* 1.) A C B E. are parallel. Therefore A C E B is a Parallelogram and conſequently C E, which hath been already ſhewn t

be perpendicular to both Planes, is equal (*per* 34. *l.* 1.) to A B. In the ſame manner I might ſhew that all the Perpendiculars to both Planes are equal. Therefore (by *Defin.* 8. *l.* 11.) the Planes are parallel. *Q. E. D.*

PROP. XV. Theorem.

IF two right Lines (B A, C A) touching each other to be parallel to two right Lines which alſo touch one another (E D, F D;) the Planes likewiſe which are drawn through them will be parallel. Fig. 22.

From A, let there be drawn A G, perpendicular to the Plane E F, and let G H, G I, be parallel to D E, D F. Theſe (*per* 9. *l.* 11.) will alſo be parallel to A C, A B. Seeing therefore the Angles I G A, H G A, be, by *Def.* 3. *l.* 11. right; C A G, B A G, will alſo (*a*) be right Angles. Therefore, G A, which is perpendicular to the Plane E F, will alſo be perpendicular to the Plane B C (*b*.) Therefore the Planes B C, E F, are, by the foregoing, parallel. *Q. E. D.*

(a) *Per* 27. *l.* 1.
(b) *Per* 4. *l.* 11.

PROP. XVI. Theorem.

A Plane (E H F G,) cutting parallel Planes (A B, C D,) makes the Sections in them, (E H, G F) parallel. Fig. 23.

If not, ſeeing they be in the ſame interſecting Plane, they will meet ſomewhere, by *Schol. Prop.* 21. *l.* 1. as in I. Wherefore ſeeing the whole Lines H E I, F G I, be in the Planes * A B, C D, produced, theſe Planes alſo will meet in I. Which is abſurd, and contrary to *Defin.* 8. *l.* 11.

* *Per* 1. *l.* 11.

PROP. XVII. Theorem.

Fig. 24. *Parallel Planes cut right Lines (B D and G H) proportionally.*

Let the right Lines B H, G D, be drawn in the Planes P V, T Q; and likewise let B G be drawn meeting the Plane R S in F, and let F C, F I be join'd. The Plane of the Triangle B G D cutting parallel Planes, makes the Sections C F, D G parallel, by the foregoing. Therefore B C is to C D, as B F (*e*) to F G. Again, the Triangle B H G cutting parallel Planes, makes the Sections, by the foregoing, B H, F I, parallel. Therefore H I is to I G, as * B F to F G; that is, as I have already shew'd, as B C is to C D. *Q. E. D.*

(e) *Per* 2. *l.* 6.

* *Per* 2. *l.* 6.

PROP. XVIII. Theorem.

Fig. 25. *If a right Line (F E) be perpendicular to a Plane (A B;) all the Planes which are drawn through it are perpendicular to the same Plane (A B.)*

Let the Plane G C be drawn through F E, making C D the common Section with A B; and let the Lines H K be drawn in the Plane G C, perpendicular to the common Section, C D. Now seeing, by the Construction, H K is perpendicular to the same common Section to which F E is perpendicular, by the Hypothesis, K H and F E must be parallel, by 29. *l.* 1. Therefore H K is also perpendicular to the Plane A B, *per* 8. *l.* 11. Therefore the Plane G C is perpendicular to the Plane A B, *per Definition* 4. *l.* 11.

PROP. XIX. Theorem.

Fig. 26. *If two Planes (M F, G D) cutting each other, be both perpendicular to the same Plane (A B;) their common Section also will be perpendicular to that Plane (A B.)*

For

For ſeeing, by the Hypotheſis, the Plane M F is perpendicular to the Plane A B; it is manifeſt, by *Definition* 4, that there may be drawn in the Plane M F, from the Point L, a Perpendicular to the Plane A B; namely, that which from L, in the Plane M F, is perpendicular to the common Section E F. Again, by the Hypotheſis, G D is perpendicular to that Plane A B; 'tis evident, in the Plane G D, may be drawn from the Point L, a Perpendicular to the Plane A B. But from the Point L (*a*) there can be erected only one Perpendicular to the ſame Plane A B. Therefore the Perpendicular to the Plane A B, which is drawn from the Point L, muſt be found in both the Planes, M F and G D, and conſequently L K, the common Section of thoſe two Planes, M F and G D, is perpendicular to the Plane, A B. *Q. E. D.*

(a) *Per* 13. *l.* 11.

PROP. XX. Theorem.

IF a ſolid Angle (A) is contain'd under three plain Angles (B A C, C A D, D A B;) any two of theſe is greater than the third.

Fig. 27.

If the three Angles be equal, the Aſſertion is manifeſt at firſt Sight; and it is as certain, if they be unequal. For let B A D be the greateſt; and from B A D, cut off B A E, equal to B A C, and make the Line A C equal to A E. And through E, let there be drawn a right Line meeting A B and A D, in B and D, and let B C, D C, be join'd. Becauſe, by the Conſtruction, the Angles B A E, B A C, are equal, as likewiſe the Sides B A, A E, equal to the Sides B A, A C, the Baſes alſo B E, B C, will be equal (*b.*) And becauſe B C, C D, (*c*) are greater than B D, the Equal, B E, B C, being taken away, there remains C D greater than E D. But the Sides E A, A D, are, by the Conſtruction, equal to the Sides, C A, A D. Therefore the Angle (*d*) C A D is greater than the Angle E A D. Seeing therefore the Angle B A C is equal, by the Conſtruction, to the Angle B A E, thoſe two Angles together, B A C, C A D, are greater than the Whole, B A D. *Q. E. D.*

(b) *Per* 4. *l.* 1.
(c) *Per* 20. *l.* 1.
(d) *Per* 25. *l.* 1.

PROP. XXI. Theorem.

THE plain Angles conſtituting any ſolid Angle whatſoever, are leſs than four right ones.

Fig. 28. Let A be a ſolid Angle; let the right Lines, B C, C D, D E, E F, F B, be ſubtended to the plain Angles which make up the ſolid one, ſo that thoſe right Lines be all in one Plane. Which being done, there is conſtituted a Pyramid, whoſe Baſe is the Polygon B C D E F; A is the Top and it is contain'd under ſo many Triangles, G, H, I, K, L, as there are plain Angles which compoſe the ſolid Angle A. And now, becauſe the two Angles A B F, A B C, are by the foregoing, greater than the third, F B C; and the two, A C B, A C D, are greater than the third, B C D and ſo on: All the Angles of the Triangles, G, H, I, K, L, about the Baſe, as taken together, are greater than all the Angles of the Baſe, B, C, D, E, F, taken together. But the Angles of the Baſe, together with four right ones, make twice ſo many right Angles, by *Theorem* 2. *Schol.* after 32 *L.* 1. as there are Sides, or, which is the ſame, as there are Triangles. Therefore all the Angles of the Triangles about the Baſe, together with four right ones, make more than twice ſo many right Angles as there are Triangles. But the ſame Angles about the Baſe, together with the Angles that compoſe the Solid make up (*a*) twice ſo many right Angles as are the Triangles. It is manifeſt therefore, that the Angles which compoſe the ſolid Angle A, are leſs than four right ones. *Q. E. D.*

(a) *Per* 32. *l.* 1.

Corollary.

FROM this and the foregoing, it is obvious to collect that a ſolid Angle may be compos'd of any three plain Angles, which are leſs than four right ones, if ſo be that any two of them be greater than the other.

Scholium.

Scholium.

FROM this Propofition is demonftrated that famous Theorem, That only three regular and equal plain Figures can contain a Body ; to wit, Equilateral Triangles, either 4, or 8, or 20 ; 6 Squares, and twelve Pentagons. And confequently, that there are only five regular Bodies. A *Pyramid*, which is contain'd under 4 ; an *Octaedrum*, which is comprehended by 8 ; and an *Icofiedrum*, which is bounded by 20 Equilateral Triangles ; a *Cube*, which is contain'd under 6 Squares ; and the *Dodecaedrum*, under 12 regular and equal Pentagons. Now a Body is called Regular, which is comprehended under regular and equal Planes.

Demonft. A folid Angle cannot be compos'd of only two Equilateral Triangles ; three, at leaft, are requir'd.

Of three Equilateral Triangles meeting in one Point, there may be conftituted the folid Angle of a *Pyramid* ; of four, the folid Angle of an *Octaedrum* ; of five, the folid Angle of an *Icofiedrum :* Forafmuch, as both 3, 4, and 5 Angles of an Equilateral Triangle are lefs than 4 right ones, as is gathered from *Corollary* 12. *Propofition* 32. *L.* 1:

And becaufe three Angles of a regular Pentagon, as is gathered from *Corollary*, *Prop.* 11. *l* 4. are lefs than four right ones, three Pentagons, meeting in one Point, will conftitute a folid Angle, that of the *Dodecaedrum*.

That of the three Squares, meeting in one Point, may be compos'd the folid Angle of a *Cube*, is manifeft of itfelf. And thus there arife five regular Bodies.

But that there are no more than thefe five, is thus proved.

Six Angles of an Equilateral Triangle make juft four right ones. For one is two Thirds of one right one ; and therefore fix fuch will make, by *Corol.* 12. *Prop.* 32. *l.* 1: twelve thirds of one right one, that is, four right ones. And therefore of fix Equilateral Triangles a folid Angle cannot be compos'd , much lefs of more.

That of four Squares a folid Angle cannot be made, much lefs of more, is manifeft in itfelf.

Four Angles of a regular Pentagon are greater than four right ones. For, by *Coroll. Prop.* 11. *l.* 4. each of them make

make ſix Fifths of one right one. Therefore a ſolid Angle cannot be made of four ſuch Pentagons ; much leſs of more.

Nor can a ſolid Angle be compos'd of any other regular Figures whatſoever. Three Angles of a regular Hexagon, by *Corollary* 2. *Prop*. 15. *l*. 4. are equal to four right ones. For one makes four Thirds of one right one ; and therefore three make twelve Thirds of one, that is, four entire right ones. Therefore of three Hexagons a ſolid Angle cannot be made up ; much leſs of more.

But ſeeing three Angles of a regular Hexagon are equal to four right ones, three Angles of any other Figure whatever greater than an Hexagon, as of an Heptagon, Octagon, *&c.* will be greater than four right ones. Wherefore it is manifeſt, that the reſt of the regular Figures are all incapable of compoſing a ſolid Angle ; and conſequently, that there can be no regular Bodies beſides the five foregoing.

PROP. XXII, XXIII.

ARE very prolix, and tedious to Beginners, and ſcarce at any Time come into Uſe.

PROP. XXIV. Theorem.

Fig. 29. *THE Planes which contain a Parallelopepid are* (1.) *Parallelograms.* (2.) *The oppoſite ones are like.* (3.) *The Planes are equal.*

Part I. The Plane A F. cutting the Planes B D, F H, which by *Defin.* 13. are parallel, makes (*a*) the Sections B A, F E. parallel. Again, the Plane A F, cutting the Planes A H, B G, which, by the ſame Definition, are parallel, (by the ſame) makes the Section A E, B F, parallel. Therefore B A E F is a Parallelogram. By the like Argument the reſt of the Parallelopepid may be prov'd to be Parallelograms.

(a) *Per* 16. *l*. 11.

Part II. Becauſe it is manifeſt from the firſt Part, that A B, B C, are parallel to E F, F G ; the Angles A B C, E F G, muſt be (*b*) equal. Wherefore ſeeing the alternate Sides

(b) *Per* 10. *l*. 11.

Sides also are equal, the opposite Parallelograms B D, F H, are like or similar. And the same of the rest.

Part III. This is manifest from the first Part, and Fourth, or Eighth of the First Book.

PROP. XXV. Theorem.

IF a Parallelopepid (G F D I) or any Prism whatever be cut by a Plane (N P) that is Parallel to the opposite Sides; there will be this Proportion, as the Base (D C P O) is to the Base (O P F E) so is the solid (G P) to the solid (N F.) Fig. 30.

This is demonstrated in the same manner as *p.* 1. *l.* 6.

Corollary.

A Prism cut by a Plane parallel to the opposite Planes, hath a Section like and equal to the opposite Planes.

PROP. XXVI, XXVII.

ARE not necessary.

PROP. XXVIII. Theorem.

A Plane passing through the Diameters of opposite Planes (A C, E G) cuts the Parallelopepid into two equal Prisms. Fig. 29

Because (*a*) B G, B E, are Parallelograms; C G, A E, are equi-distant from the same B F. Therefore (*b*) they are also parallel betwixt themselves, and consequently are in one Plane. Therefore the right Lines A C, E G, are (*c*) in one Plane. But now that a Plane drawn through them doth cut the Parallelepepid into two equal Prisms, is thus shew'd. Let the Prism A E G C D H be understood to be so constituted upon its Plane A E C G, that the Angles D, H.

(a) *Per* 24. *l.* 11.
(b) *Per* 9. *l.* 11.
(c) *Per* 7. *l.* 11.

H, bend towards the Angles B, F. It is manifeſt, that it will yet be betwixt the parallel Planes B A D C, F E H G. But then D muſt needs fall upon B, and H upon F. For let D fall without B, if it may be, and in N. The Angle B A C (*d*) is equal to the Angle D C A. But D C A is equal to N A C, for it is one and the ſame Angle. Therefore B A C and N A C are equal: Which is abſurd. Therefore D falls upon B; and for the ſame cauſe, H upon F. Therefore the Priſm A E G C D H coincides with the Priſm A C G E F B, and conſequently they are equal, by *Axiom* 7.

(d) *Per* 27. *l.* 1.

PROP. XXIX, XXX. Theorems.

Fig. 31.

THE Parallelopepids (FEAGKIMC) and (FEBHLOMI) which have the ſame Baſe (EFIM) and the ſame Altitude, and conſequently exiſt between parallel Planes (EFIM) and (GAOL) are equal.

For they either exiſt betwixt the lateral parallel Planes E A O M and F G L I, or not. Let the firſt be ſuppos'd, from the 24th of this, and the 8th of the firſt Book, it is manifeſt, that the Triangles A E B, C M O; likewiſe G F H, K I L, are Equilateral and Equiangular to each other. Wherefore, as in the foregoing, I might ſhew that the Priſms C M O L I K, and A E B H F G, being laid upon each other will coincide, and conſequently, by *Axiom* 7. are equal. Wherefore the common Solid F E B H K C M I, being added, the whole Parallelopepids F E A G K I M C and F E B H L O M I are equal. *Q. E. D.*

Then let the Parallelopepid F X Q E M I P R not exiſt betwixt the ſame lateral parallel Planes with the Parallelopiepd F E A G K C M I. Here, becauſe, by the Hypotheſis, G K, A C, R P, Q X, are in one Plane, which is parallel to the Baſe E F I M; let R P, Q X, cut G K in L and H, and A C in O and B; and let E B, M O, F H, I L, be join'd. It is eaſy now to ſhew, that the Planes containing the Solid F E B H L O M I, are Parallelograms, the oppoſite ones of which are equi-diſtant, and conſequently that the Solid is, by *Defin.* 13. *L.* 11. a Parallelopepid. But to this, by the firſt Part, the Parallelopepids F X Q E M I P R, and

and F E A G K C M I, are each of them equal. Therefore they are alfo equal betwixt themfelves. *Q. E. D.*

Corollary.

THIS Propofition is like to the 35th of the firft Book; for it affirms concerning Solids, what that doth touching Planes. Wherefore the Demonftration of the reft of the Cafes will be like alfo.

PROP. XXXI. Theorem.

Parallelopepids upon equal Bafes (AO and E G) and in the fame Altitude (S) are equal. Fig. 33.

Firft, let the Parallelopepids have their Sides perpendicu-ar to the Bafes. Unto the Side F G, produced, let there be made a Parallelogram G M K H, equal and like to the Parallelogram A O; and the Parallelogram G M P R being perfected, let the right Lines P M, R G, meet K H in Q and L. And now let Parallelopepids be underftood to be conftituted upon G K, G Q, G P, whofe Sides are perpendicular to the Bafes, and S is their common Altitude. The Solid E G S, is to the Solid G P S, as E G, *per* 25. *l.* 1. is to G P; that is, becaufe E G, A O are equal, by the Hypothefis, as A O to G P; that is, by the Conftruction, as G K is to G P; that is, as G Q is to G P, *per* 35. *l.* 1. that is, as the Solid G Q S is to the fame Solid G S P, *per* 25. *l.* 11. Becaufe therefore the Solids E G S and G Q S have the fame Proportion to the Solid G P S, the Solid E G S will be equal to the Solid G Q S; that is, to the Solid G K S, *per* 29. *l.* 11. that is, becaufe the Bafes G K, A O, are equal and like, by the Conftruction, to the Solid A O S, as it appears from 29. *l.* 11. and even in itfelf. Which was the Thing propos'd. Note, That in this reafoning, the Solids are fuppos'd to be right or perpendicular ones.

Then let the given Parallelopepids E G S, A O S, have their Sides at the Bafes E G, A O, oblique. Let there now be made upon E G, A O, Parallelopepids, whofe Sides are perpendicular to the Bafes in the Heighth S; thefe will be equal to the oblique ones by the 29th and 30th. Wherefore feeing, by the firft Part, right Parallelopepids are

are equal betwixt themfelves, the oblique ones will be equal betwixt themfelves likewife. *Q. E. D.*

PROP. XXXII. Theorem.

Fig. 34. *ALL Parallelopepids whatever of equal Heighth, are betwixt themfelves as their Bafes.*

Let G O and A be the Bafes. Upon C O make the Parallelogram O E equal to A.

Upon B C, O E, let Parallelopepids be underftood to be erected in the Altitude K; thefe therefore will be Parts of one Parallelopepid, B E K. Therefore the Parallelopepid, O E K, is to the Parallelopepid, B C K, as the Bafe O E, to the Bafe B C, *per* 25. *l.* 11. that is, by the Conftruction, as the Bafe A is to the Bafe B C. But becaufe the Bafes O E and A, are equal, the Parallelopepids, O E K and A K, are equal, by the foregoing. Therefore alfo the Parallelopepid, A K, is to the Parallelopepid, B C K, as the Bafe A is to the Bafe BC. *Q. E. D.*

Scholium.

THAT which hath here been fhew'd of Parallelopepids will be demonftrated in the Twelfth Book of Pyramids, *Prop.* 6. Of all Prifms whatever, in *Corollary* 1. after *Propofition* 9. Of Cones and Cylinders, *Propofition* 11

PROP. XXXIII. Theorem.

Fig. 35. *LIKE Parallelopepids (H A and C M) are in a triplicate Proportion of their homologous Sides (A B, B C)*

Let the Parallelopepids, A H, C M, be like. Therefore all their Planes, by *Defin.* 9. *l.* 11 are like; and confequently A E, by *Defin.* 1. *l.* 6. is to B C, as E B to B O and as F B is to B G, fo is E B to B O. Moreover the Angles of the Planes are alfo equal, by the fame. Therefore, let the Solids, A H, C M, be fo placed, that the equal Angles C B O, A B E, may be oppofite, and the Side

Sides A B, C B, may lie so as to make one strait Line; and then EB, O B will also lie so as to make one strait Line. Now, let Solids be imagin'd to be constituted upon the Planes B Q and E C, in such sort that the Solids K B. H A, may be one Parallelopepid, and K B, P O, may make one Parallelopepid, and P O, C M. may make one Parallelopepid likewise. The Solid H A, is to the Solid K B, *per* 25. *l.* 11. as A E to B R; that is, *per* 1. *l.* 6. as A B to B C; that is, as I shew'd above, by the Hypothesis, as E B is to B O; that is, by the same, as E C is to B Q; that is, *per* 25. *l.* 11. as the same Solid K B, is to the Solid P O. Therefore the three Solids, H A, K B, PO, continue the same Proportion, But now the Solid K B, is to the Solid P O, by the same, as the Base B R, is to the Base B Q; that is, *per* 1. *l.* 6. as E B is to B O; that is, as F B is to B G, as it was shew'd above, by the Hypothesis; that is, by the same, as the Plane F C is to the Plane B S; that is, *per* 25. *l* 11. as the same Solid P O again is to the Solid C M. Therefore the four Solids, H A, K B, P O, C M, are continually proportional. Therefore, by *Defin.* 10. *l.* 5. the Proportion of the first H A, to the fourth C M, is triplicate of the Proportion of the first H A, to the second K B; that is, triplicate to the Proportion, *per* 25. *l.* 11. of A E to B R; that is, triplicate, *per* 1. *l.* 6. to the Proportion of the homologous Sides, A B to B C. Q. *E. D.*

[Corollary (1.) " Hence, if there be four right Lines ' continually proportional; as is the first to the fourth, ' so is a Parallelopepid describ'd upon the first. to a Parallelopepid like, and in like manner describ'd upon the ' second.

(2.) " Upon this also depends that most famous Problem concerning doubling the Cube; of which afterwards, *Scholium*, *p.* 18. *l.* 12.

(3.) " Hence also is to be corrected the Error of those, ' who suppose that the Proportion of like Solids is the ' same as is that of their Sides. For the Cube of a Line, ' which is double to another Line, is not only double to ' the other, but as eight to one. And the Cube of a ' Line, which is treble to another Line, is not only treble ' to the other Cube,. but contains it 27 Times. For 1 : ' 2 : 4 : 8 ÷ and 1 : 3 : 9 : 27 ÷, and the same thing is

" to be ſaid of all like Bodies whatſoever; as will appear " afterwards.

(4.) " Hence the triplicate Proportion of any Quantities whatſoever is the Proportion of the Cubes of the " ſame Quantities. Let there be any two Quantities in the " triplicate Proportion of the Quantities, A B, B C; they " ſhall alſo be as A B Cube, is to B C Cube.]

Scholium.

THAT which hath here been ſhew'd of Parallelopepids, will be demonſtrated, Book 12. Of Pyramids, *Prop.* 8. Of all Priſms whatſoever, *Corollary* 2. *Prop.* 9. Of Cones and Cylinders, *Prop.* 12. Of Spheres, *Prop.* 18.

PROP. XXXIV. Theorem.

Fig. 36. *IF the Parallelopepids (B M, C K) be equal, their Baſes and Altitudes are reciprocally proportional; (that is, the Baſe A M is to the Baſe F K, as reciprocally the Heighth F C is to the Heighth A B.)*

And if their Baſes and Altitudes be reciprocally proportional, they are equal.

Part I. Firſt, let the Sides be perpendicular to the Baſes If now the Altitudes of the Solids, B M, C K, be equal the thing is manifeſt.

If the Altitudes be unequal, from the greater, F C, cu off F E, equal to B A; and through E draw the Plane E L parallel to F K. The Baſe A M, is to the Baſe F K, *pe* 25. *l.* 11. as the Solid B M, is to the Solid E K; that is becauſe, by the Hypotheſis, the Solids B M, C K ar equal, as the S lid C K, is to the Solid E K; that is, b the ſame, as C G is to E G; that is, *per* 1. *l.* 6. as C F i to E F; that is, by the Conſtruction, as C F to B A *Q. E. D*

Then let the Sides be oblique to the Baſes. Let rig Parallelopepids be erected upon the ſame Baſes in the ſam Height. The oblique Parallelopepids will be equal to theſ Wherefore ſeeing theſe, by the firſt Part, have their Baſ and Altitudes reciprocal, thoſe alſo will be likewiſ *Q. E. D.*

Th

Then let the Sides be oblique to the Bases. Let right Parallelopepids be erected upon the same Bases in the same Height. The oblique Parallelopepids will, *per* 29 and 30. *l.* 11. be equal to these: Wherefore seeing these, by the first Part, have their Bases and Altitudes reciprocal, those also shall be so likewise. *Q. E. D.*

Part II. Let the Altitudes be unequal, and the Sides perpendicular to the Bases; and from the greater, C G, take E F, equal to A B. The Solid B M, is to the Solid E K, *per* 32. *l.* 11. as A M is to F K; that is, by the Hypothesis, as C F is to A B; that is, by the Construction, as C F is to E F; that is, as C G is to (*a*) E G; that is, (*b*) as the Solid C K is to the same Solid E K. Therefore the Solids B M and C K have the same Proportion to E K: Therefore they are equal. *Q. E. D.*

(a) *Per* 1. *l.* 6

(b) *Per* 25. *l.* 11.

Corollaries.

WHAT Affections have been demonstrated of Parallelopepids, *Prop.* 29, 30, 31, 32, 33, 34, do also agree to Triangular Prisms, which are the Halves of Parallelopepids. As is manifest from *Prop.* 28. Therefore,

1. The Triangular Prisms, which are of equal Heighth, are as their Bases A, B. *Fig.* 37.

2 If they be like, their Proportion is triplicate to the Proportion of the Sides, opposite to the Angles.

3. If they be equal, they reciprocate their Bases and Altitudes; and if they reciprocate their Bases and Altitudes, they are equal.

Scholium.

WHAT hath here, in *Prop.* 34. been shew'd of Parallelopepids, will be demonstrated in the 12th Book of Pyramids, *Prop.* 9. Of all Prisms whatsoever, *Corollary* 3. after *Prop.* 9. Of Cones and Cylinders, *Prop.* 15.

PROP. XXXV.

IS very long, and subservient to the following Proposition, which we will demonstrate without it.

PROP. XXXVI. Theorem.

Fig. 38. *A Parallelopepid (D H) made of three proportional right Lines (A, B, C,) is equal to a Parallelopepid (I N,) which is made of the Mean (B,) and is Equiangular to the former.*

Let the Baſe F D, of the Parallelopepid D H, have the Side E F equal to A, and the other Side E D equal to C: And the Side E G, which ſtands upon the Baſe equal to B. Thus the Parallelopepid D H will be made of the three right Lines, A, B, C. Then let the three Sides, L X, I X, X M, and conſequently all the reſt, of the Parallelopepid I N be equal to the middle Line B: And the ſolid Angle X, equal to the ſolid Angle E; the Parallelopepid I N will be made of the Mean B, and be Equiangular to the former. I ſay alſo that it is equal.

For, ſeeing, by the Hypotheſis and the Conſtruction, as F E is to L X, ſo reciprocally, I X is to D E, the Baſes alſo (a) D F, I L, will be equal. Now, becauſe the ſolid Angles at E and X are equal; if they be put within one another, (b) they will coincide; and becauſe of the Equality of the right Lines, E G, X M, the Points M and G, will coincide. Wherefore both the Solids will have one perpendicular Altitude; to wit, the right Line, which is let fall from the Points M, G, now become one, unto the Plane of the Baſe. The Solids therefore D H, I N, * are equal. *Q. E. D.*

(a) *Per* 14. *l.* 6.
(b) *Per Defin.* 11. *l.* 11.
* *Per* 31. *l.* 11.

Scholium.

WE will further obſerve what is of great Uſe, that of three Lines drawn into or multiplied one by another, after what manner ſoever, a Solid of the ſame Magnitude is produced.

A B C. C A B. B C A.
1. 2. 3.

In the preſent Scheme, the two firſt Letters deſign the Baſe; the third, the Altitude. Let us compare the firſt with the ſecond.

The Baſe A B is to the Baſe C A, *per* 1. *l.* 6. as the Side B is to the Side C; that is, reciprocally, as the Heighth B is to the Heighth C. Therefore, by *Prop.* 34.

A B C,

A B C, is equal to C A B:

In the ſame manner it may be ſhew'd that the firſt is equal to the third, and the third to the ſecond.

PROP. XXXVII. Theorem.

PArallelopepids which are like, and deſcribed in the like manner by proportional right Lines, will themſelves alſo be proportional; and converſly.

This is manifeſt of itſelf. For the Proportions of Parallelopepids, by the 33d of this Book, will be triplicate to thoſe Proportions which, by the Hypotheſis, are equal, which the Lines have betwixt themſelves.

The Converſe is manifeſt of itſelf alſo.

The Propoſition is true of all ſorts of like Bodies, which will appear from Book the 12th, to have betwixt themſelves a Proportion triplicate to that which the Sides have.

PROP. XXXVIII, XXXIX. Theorems.

THESE contain nothing remarkable, and are ſcarce of any Uſe

PROP. XL. Theorem.

THIS is of a ſmall Uſe, and indeed no other than the 28th Propoſition in another View.

Scholium.

FROM what hath hitherto been demonſtrated, we have the Dimenſion of Triangular Priſms, and of Quadrangular or Parallelopepids; to wit, if the Altitude be multiplied into the Baſe. As if the Altitude be of 10 Feet, and the Baſe of 100 ſquare Feet, (now the Baſe is meaſured by *Scholium, p. 36*, or 41. *l.* 1.) multiply 10 by by 100, there will ariſe 1000 Cubic Feet for the Solidity of the given Priſm.

The Demonſtration is eaſy. For, like as a Rectangle ariſeth from the Multiplication of one Side by another, ſo a right Parallelopepid is produced from the Heighth drawn into the Baſe Therefore every Parallelopepid is alſo produced from the Altitude multiply'd into the Baſe; ſeeing by 31. *l.* 11. it is equal to a right Parallelopepid, conſtituted upon the ſame Baſe with the ſame Heighth.

Then ſeeing the whole Parallelopepid is produced from the Heighth into the whole Baſe; the half of the Parallelopepid (that is, a Triangular Priſm by 28. *l.* 11.) will be produced from the Altitude multiplied by half the Baſe; *Fig.* 29. to wit, the Triangle I L K.

T H I

THE

Elements of EUCLID.

BOOK XII.

With Us the Eighth.

WHAT in the foregoing Books we have endeavoured to perform; namely, to bring the Elements of the Mathematicks into a more easy and brief Method, will be to be endeavour'd in this Twelfth Book especially; the Doctrine whereof is most necessary, but the Demonstrations are so prolix, that they commonly make Beginners almost to despair. We have so propos'd to our selves to remedy this Evil, that in the mean while we will not depart from the Rigour of Geometrical Demonstration. Which Thing, whether or no we have attain'd, the Reader will understand, if he shall compare this of ours with *Euclid*'s Prolixity.

" Now, after *Euclid* had in the former Book declared
" the Elements of Solids, and defined the Measures of the
" most easy Bodies, those, namely, which are terminated
" with plain Surfaces: In this Twelfth Book he considers
" Bodies bounded with curve Surfaces; to wit, Cylinders,
" Cones and Spheres; compares them betwixt themselves;
" and defines their Measures. This Book is indeed most
" profitable, because it contains those Principles on which
" the chief Masters of Geometry, and especially *Archimedes*, have built so many famous Demonstrations, con-
" cerning the Cylinder, Cone and Sphere.

DEFINITIONS.

*Fig.*1. *l.*12. 1. A Pyramid is a Solid [Z L] comprehended under the Triangles [ALC], CLF, FLB, BLA] placed from one Plane [Z] to one Point [L.]

The Plane Z is called the Baſe, and may be either a Triangle or Quadrangle, or any other Figure, from each of the Sides whereof there ariſe Triangles meeting together in the Point L, which is called the Vertex, or Top.

As the Triangle amongſt Rectilinear plane Figures, ſo the Pyramid amongſt ſolid ones is the firſt and moſt ſimple.

Fig. 2, 3. 2. If without the Plane of ſome Circle [C L] there ſhall be taken the Point [A,] and from it be drawn the infinite right Line [A F,] touching the Circle in C; and this Line (the Point [A] remaining fix'd) be turn'd about the Circumference of the Circle, until it returns thither, from whence it began to be moved; the Surface deſcribed by the right Line [A C F] is term'd a conical Surface; and the Body, which is contain'd under this Surface, and the Circle [C L] is call'd a Cone.

The Vertex of the Cone is [A.]

The Circle [CL] is the Baſe of the Cone.

The right Line [A B,] drawn from the Vertex to the Center of the Baſe, is the Axis of the Cone.

The Side of the Cone is the right Line [A C drawn from the Vertex to the Circumference of the Baſe, which that it is wholly in the Surface of the Cone, is manifeſt from the Production of the Figure.

* *Fig.* 2. A right * Cone is, when the Axis [A B] is perpendicular to the Baſe.

† *Fig.* 3. A ſcalene † or oblique Cone, is, when the Axis [A B] is not perpendicular to the Baſe.

A right Cone is alſo made by a right-angled Triangle [CBA] turn'd round about one of the perpendicular Sides [A B.] See *Fig.* 2.

Fig. 4, 5. 3. If an infinite right Line [C O F] be turn'd about, two Circles [C L, O Q] equal and parallel, until it returns to that Place from whence it began to be mov'd, and remains always, whilſt it is mov'd, parallel to it ſelf, the Surface deſcribed by the right Line [C O F] is called a Cylindrical Surface; and the Body which is contain'd under this Surface, and the two Circles, is call'd a Cylinder.

The

The Bases of the Cylinder are the Circles [C L, O Q] the right Line [A B] which connects the Centers of the Bases, is called the Axis. The right Line [O C] in the Surface of the Cylinder, touching both the Bases, is called a Side of the Cylinder.

A right Cylinder, is, when the Axis is perpendicular to the Base. *Fig.* 4.

A scalene or oblique Cylinder, is, when the Axis is not perpendicular to the Base. *Fig.* 5.

A right Cylinder, is also made by a Rectangle [O C B A] turn'd round about one Side [B A.] See *Fig.* 4.

4. Like Cones and Cylinders are those, which have their Axes [A K, Z O] and the Diameters of their Bases [B F, Q R] proportional. *Fig.* 20, 21.

5. A Sphere, is a Solid contain'd under one Surface, into which Surface all the right Lines that are drawn from a certain Point within the Figure, are equal amongst themselves. That Point is call'd the Center. The Diameter of the Sphere is a right Line drawn through the Center into the Surface on both Sides.

A Sphere is produced if a Semi-circle be turn'd about its Diameter [A F] which remains in the mean while unmov'd. *Fig.* 6.

6. Magnitudes inscrib'd in, or describ'd about some Figure, whether they be greater or lesser than the Figure, are then said to *end* in the Figure, when they will at the last differ from it by a Quantity less than any given one whatsoever, or how small soever.

Therefore if those Magnitudes which are inscrib'd into some Figure will at last fall short of it by a Deficiency less than any given one whatsoever, the Magnitudes inscrib'd are said to *end* in the Figure; and if those which are circumscrib'd about some Figure, will at last exceed it by an Excess less than any given one whatsoever, they shall be said to end in the Figure.

PROP.

PROPOSITION I. Theorem.

Fig. 6, 7. *THE Proportion of like Polygons inscrib'd in a Circle, is duplicate to the Proportion of the Diameters (AF, IC.)*

Let A O, B F; I R, L C be drawn. Because the Polygons are suppos'd to be like, the Angles (O B A, R L I) will (*per Defin.* 1. *l.* 6.) be equal; and the Sides O B, B A, proportional to the Sides R L, L I. Therefore in the Triangles A O B, R I L (*per* 6. *l.* 6.) the Angles O and R are equal. Therefore also the Angles B F A and L C I, which stand upon the same Arches, B A, L I, are (*per* 21. *l.* 3.) equal. But the Angles, F B A, C L I, in Semi circles, are (*per* 31. *l.* 3.) right ones. Therefore the other Angles (*per Corol.* 9. *p.* 32. *l.* 1.) B A F, L I C, are equal. Therefore because the Triangles F A B, C I L, are Equiangular to each other, they are (*p.* 4. *l.* 6.) like: and B A will be to L I, as A F to I C. Now, because, by the Hypothesis, the Polygons are like, their Proportion will be duplicate (*p.* 20. *l.* 6.) to the Proportion of the Sides B A, L I, that is, as I have already shew'd, duplicate to the Proportion of the Diameters A F, I C. *Q. E. D.*

Corollary.

Fig. 6, 7. THE Circumferences of like Polygons inscribed in a Circle are betwixt themselves as the Diameters.

Seeing it hath already been shew'd, that A B is to L I as A F is to I C; O B will also be to R C, as A F to I C: And so of the rest of the Sides. Therefore all the Sides together will be to all the Sides together, that is one Circumference to another, as A F is to I C.

A Lemma.

Fig. 8. POLYGONS inscrib'd in a Circle, end in a Circle. Inscribe a Square, as A C B D. Seeing this is half (*per Schol. p.*6. and 7. *l.* 4.) of the Square which is circumscrib'd it will be greater than half of the Circle. Wherefore i[n]

th[e]

this be taken out of the Circle, there will be taken out of it more than half. Then each Arch being bisected in E, K, I, H, inscribe an Octagon : And let F G touch the Circle in E ; which F G, let B C, D A meet in G and F ; C F will be a Parallelogram, of which, seeing the Triangle C E A (*per* 41. *l.* 1.) is half, this will be more than half of the Segment C E A. In the same manner each of the Triangles, A K D, D I B, *&c.* is more than half each of the Segments. Therefore all the Triangles are more than half all the Segments. Therefore if you take these out of those, that is, out of the Remainder of the Circle, more than half will be taken away. In the same way of arguing, if there be inscrib'd in the Circle, Polygons of Sides always double in Number ; I can shew that there will always be taken out of the Remainder of the Circle more than half. Therefore the Remainder must at last be less than any given one whatsoever ; and consequently the inscrib'd Polygons will at last fall short of a Circle by a Quantity less than any given one whatsoever ; that is, (*per Defin.* 6. *l.* 12.) will end in a Circle.

PROP. II. Theorem.

THE Proportion of Circles is duplicate to the Proportion of their Diameters. Fig. 6, 7.

The Proportion of Polygons inscrib'd in a Circle without End, is (*per* 1. *l.* 12.) duplicate to the Proportion of the Diameters. But Polygons (by the foregoing *Lemma*) inscrib'd in a Circle infinitely, at last end in the Circle. Therefore the Proportion of Circles is also duplicate to the Proportion of the Diameters.

PROP. III, IV.

ARE prolix, and hard for young Beginners, and have no other Use, than that they serve to the Demonstration of the Fifth, which we shall demonstrate much more easily without them.

Lemmata,

Lemmata, or preparatory Propositions to Prop. V.

Lemma I.

Fig. 9. IF two Triangular Pyramids be cut with Planes (O S E, R X Z) parallel to the Bases (A B C, I Q V) which same Planes divide the Sides (C F, Q L) proportionally in (E and Z, then O S E, R X Z) will be betwixt themselves, as the Bases (A C B, I Q V.)

Because the parallel Planes, O S E, A B C, are cut by the Planes B F C, A F B, A F C. the common Sections, S E, B C, and O S, A B, and O E, A C, (will be *per* 16. *l.* 11.) parallel. Wherefore the Angles O S E, A B C, and S O E, B A C, and O E S, A C B two and two, are (*per* 10. *l.* 11.) equal. Wherefore the Sections, O S E, A B C, are like (*per* 4. *l.* 6.) In the same manner I might shew that the Sections R X Z, I V Q, are like. Therefore (*per* 19. *l.* 6) the Proportion of the Section A B C, to the Section O S E, is duplicate to the Proportion of the Side B C, to the Side S E; and the Proportion of the Section I V Q to R X Z, is duplicate to the Proportion of V Q to X Z. But the Proportions of B C to S E, and of V Q to X Z, are the same (for B C is to S E (by *Corollary* 1. *per* 4. *l.* 6) as C F to E F; that is, by the Hypothesis, as Q L to Z L; that is, (by the same *Coroll.*) as V Q to X Z. Therefore the Proportion of A B C to O S E, is the same with the Proportion I V Q to R X Z. *Q. E. D.*

Lemma II.

Fig. 10. PRISMS inscrib'd infinitely in a Pyramid (Z C A F) which hath a Triangular Base, end in the same Pyramid.

Let the Side of the Pyramid be divided into a certain Number of equal Parts, A B, B G, G F, and through B and G, there being made the Sections, G D N and B E P, parallel to the Base Z A C; let the Triangular Prisms B E P M A O and G D N K B Q be understood to be inscrib'd in the Pyramid. These then being continued without the Pyramid, let there be understood to be describ'd about the Pyramid the Prisms C I B A, P X G B, N H F G. The Excesses of the circumscrib'd Prisms above the inscribed ones, are the Solids I M, X K, H G, which, taken together,

gether, are equal to the Prifm C I B A : For H G (*per* 25. *l.* 11.) is equal to D B; and confequently H G with X K, are equal to P X G B, that is, (by the fame) to M E B A. Therefore the three, H G, X K, I M, are equal to the Whole, C I B A. But if A F be divided without End into more equal Parts, and confequently the Number of Prifms be infinitely encreafed, A B will become lefs than any given Line. Therefore (as it is manifeft from *p.* 25. *l.* 11.) the Prifm C I B A will become lefs than any given one. Therefore the Excefs of their circumfcrib'd Prifms, (and much more of the Pyramid Z C A F, which is part of the Prifms circumfcrib'd about it) above the infcribed Prifms, will be lefs than any given Prifm. Therefore the infcribed Prifms (by *Defin.* 6. *l.* 12.) end at laft in a Pyramid. *Q. E. D.*

PROP. V. Theorem.

TRiangular Pyramids of the fame Heighth have that Proportion betwixt themfelves, which their Bafes (A Q R, E S X) have.

Let the equal Altitudes of the Pyramids be reprefented by the Side A P, E Z; which, on both Sides, let be divided into as many equal Parts as you will, but fo that they be of the fame Number; and let there be made through the Points of the Divifions, Sections parallel to the Bafes: Let Triangular Prifms, of the fame Number, and the fame Heighth, be underftood to be infcrib'd in both Pyramids. And now becaufe the Prifms, L A, I E, are of the fame Heighth, the Prifm L A will be to the Prifm I E (by *Coroll.* 1. *p.* 34. *l.* 11.) as the Bafe L O B is to the Bafe I N K; that is, by (*Lemma* 1.) as the Bafe Q R A is to the Bafe S X E. In the fame manner I might fhew that each of the Prifms infcrib'd in the Pyramid Q P A R, is to each infcrib'd in the Pyramid S Z E X, as the Bafe Q A R is to the Bafe S E X. Therefore all of them together, are to all of them together, as Bafe is to Bafe. Wherefore feeing they at laft end (*per Lem.* 2.) in the Pyramids themfelves, the Pyramids themfelves alfo will be as their Bafes. *Q. E. D.* *Fig.* 11.

PROP.

PROP. VI. Theorem.

Fig. 12, 13. *ALL Pyramids whatſoever, which are of equal Heighth, have that Proportion betwixt themſelves, which their Baſes (AB, CFO) have.*

Let their Baſes be reſolv'd into Triangles, A, B, C, F, O: and the whole Pyramids into Triangular Pyramids. The Pyramid A X. is to the Pyramid O Z (by the foregoing) as A is to O; and the Pyramid B X, is to the Pyramid O Z, as B is to O (by the ſame.) Therefore the Pyramids A X, B X, together (that is, the whole Pyramid A B X) are to the Pyramid O Z, as A, B, together, are to O, By the ſame Argumentation, the Pyramid A B X, is to the Pyramid F Z (by the foregoing) as A, B. are to F: And A B X, is to C Z, as A, B, is to C. Therefore A B X, is to the three, O Z, F Z, C Z, together; that is, to the whole Pyramid, O F, C Z, as, A, B, together, is to O, F, C, together. *Q. E. D.*

PROP. VII. Theorem.

Fig. 14. *EVERY Pyramid is the third Part of a Priſm, which hath the ſame Baſe and Heighth.*

Firſt, let the Triangular Pyramid, B G A C, have the ſame Baſe and Heighth with the Priſm, B A C F E O: Let B F, A O, A F, be drawn The Triangles, B F C, B F O are (*per* 34. *l.* 1.) equal. Therefore the Pyramid, B F C A, is equal to the Pyramid, B F O A For the ſame Reaſon, O E A F, is equal to the Pyramid. O B A F; that is, to the Pyramid, B O F A, for they are the ſame Pyramids Therefore B F C A, and O E A F, are alſo equal. Therefore all three. B F C A, O E A F, O B A F, or B O F A are equal. Therefore the three together are triple of one B F C A. But thoſe three conſtitute the Priſm, B A C F E O That Priſm therefore is triple to the Pyramid, B F C A that is, (*per* 5. *l.* 11.) to B G A C. *Q. E. D.*

Then let any Pyramid whatſoever have the ſame Baſ and Heighth with the Priſm, A E F H: The Lines B C B O, B E, and N I, N G, N H, being drawn, reſolve th
Fig. 15. Priſms into Triangular Priſms, and the Pyramid into Tri angula

angular Pyramids. Which being done, the Demonſtration is manifeſt from the firſt Part: For each Part of the Priſms will be triple of each Part of the Pyramids. And conſequently the whole Priſm will be triple to the whole Pyramid. *Q. E. D.*

PROP. VIII. Theorem.

THE Proportion of like Pyramids (OACB, KHIN) is triplicate to that which the homologous Sides (AB, HN,) have to each other.

Firſt, let them be Triangular: The Parallelograms, AM and HQ being perfected, ſet upon them the Parallelopepids, AG, HL, in the Heighth of the Pyramids; which ſeeing the Pyramids are like, will alſo (as appears from *Defin.* 9. *l.* 11. be like. Then let EF, RP, be drawn; and through EF, CB, as likewiſe through RP, IN, the Parallelopepid will be cut (*per* 28. *l.* 11: into two equal Priſms; each of which will be triple to the Pyramids, OACB and KHIN (by the foregoing). Therefore both together, that is, the whole Parallelopepids, AG, HL, will be ſix-fold of the Pyramids. Therefore the Pyramids are proportional to the Parallelopepids. But (*per* 33. *l.* 11.) the Proportion of theſe each to other is triplicate to the Proportion of the Sides, AB, HN. Therefore ſo likewiſe is the Proportion of the Pyramids. *Fig.* 16.

But if the like Pyramids ſhall be polygonal, let them be reſolv'd into the Triangular ones, AR, BR, CR and OK, EK, FK. You may from 20. and 5. *l* 6. and *Defin.* 9. *l.* 11. eaſily ſhew, that AR is like to OK, and BR to EK, and CR to FK. Therefore, by the former Part, the Proportion of the Pyramids, AR, OK, is triplicate to the Proportion of IM to PZ: And the Proportion of the Pyramids, BR and EK, is triplicate to the Proportion of MX to SZ; that is, again, by the Hypotheſis, of IM to PZ; and the Proportion of the Pyramids, CR, FK, is triplicate to the Proportion of XQ to ST; that is, again, of IM to PZ. Seeing therefore the Proportion of each to each is triplicate to the Proportion of IM to PZ, the Proportion alſo of all to all (that is, the Proportion of the whole Pyramid, ABCR, to the whole, OEFK) will be triplicate to the Proportion of IM to PZ. *Q. E. D.* *Fig.* 17.

PROP.

PROP. IX. Theorem.

Fig. 18, 19. *EQUAL Pyramids have their Bases and Altitudes reciprocally proportional; and those which have them so, are equal.*

Part I. First, let the Pyramids be Triangular, B A C O, K H N L: The Parallelograms B E, H R, being perfected, upon these set the Parallelopepids, B F, H P. These will be (as was shew'd in the foregoing) six-fold of Pyramids, which are, by the Hypothesis, equal, and consequently will be equal betwixt themselves. But now the Altitudes of these Parallelopepids H K, B A, are the same with those of the Pyramids, and the Bases B E, H R, are double to the pyramidal Bases, (*per* 34. *l.* 11.) B C O, H N L, and consequently proportional to them. Seeing therefore by reason of the Equality of the Parallelopepids, as B E is to H R, so (by the same) is reciprocally H K, to B A; it will also be that, as the Base B C O is to the Base H N L, so, reciprocally, is the Altitude H K to the Altitude B A. *Q. E. D.*

But if the Pyramids have polygonal Bases, let them be reduced into Triangular ones, retaining the same Altitudes; and these will be equal to those by the sixth. But the Pyramids thus reduced have, as we have now demonstrated, their Bases and Altitudes reciprocally proportional. Therefore the given polygonal Pyramids also have their Bases and Altitudes reciprocally proportional. *Q. E. D.*

Part II. Because it is now supposed, that B C O is to H L N, as H K is to B A; B E will also be to H R, as H K is to B A. Therefore the Parallelopepids, B F, H P, are (*per* 34. *l.* 11.) also equal. Therefore their sixth Parts also, to wit, the Pyramids B A C O, H K N L, are equal. *Q. E. D.*

Corollaries.

WHAT has been demonstrated of Pyramids in *Proposition* 6, 8, 9. does also agree to all Prisms whatsoever; seeing these are (*per* 7. *l.* 12.) triple to Pyramids which have the same Bases and Altitudes. Therefore,

1. In Prisms of the same Heighth, their Proportion is the same as that of their Bases. For this was shew'd of Pyramids, *Prop.* 6.

2. The

2. The Proportion of like Prifms is triplicate to the Proportion of their homologous Sides. For this was fhew'd concerning Pyramids, *Prop.* 8.

3. Equal Prifms have their Bafes and Altitudes reciprocally proportional; and thofe which have them fo are equal. For this is fhew'd of Pyramids, *Prop.* 9.

It is ftrange that thefe Things were pafs'd over by *Euclid,* feeing they are the chief Things which can be delivered concerning Rectilinear Solids.

Scholium.

FROM what has been hitherto demonftrated is drawn the Method of meafuring any Prifms or Pyramids whatfoever.

The Solidity of a Prifm is produced from the Altitude multiplied into the Bafe; and that of a Pyramid from the third Part of the Altitude multiplied by the Bafe.

As if the Altitude of a Prifm be of 5 Feet, but the Bafe contains 25 fquare Feet; multiply 25 by 5, and there arife 125 cubic Feet for the Solidity of the Prifm.

For let there be a polygonal Prifm, as A H. And let the Triangle B A C be underftood to be equal to its Bafe A E, and upon B A C, the Prifm B E to be fet at equal Heighth with A H. The Prifms B E, A H will be (by *Corollary* 1. foregoing) equal. But the Prifm B E (by *Schol. p.* 40. *l.* 11) is produced from its Altitude drawn into the Bafe B A C; that is, into A E, by Conftruction. Therefore the Prifm A H alfo is made of its Bafe A E, multiplied by its Heighth, which is fuppofed to be equal to the Heighth of the Prifm B E. *Fig.* 15, 16.

From hence, and from the 7th, the Demonftration of the fecond Part is alfo manifeft.

A Lemma to Propofition 10.

PYRAMIDS and Prifms, which are infcrib'd in Cones and Cylinders infinitely, do at laft end in the Cones and Cylinders.

This is demonftrated as the *Lemma* of *Propofition* 2 with the help of *Propofition* 6. and of *Corollary* 1. after *Propofition* 9. if as their Planes infcrib'd in a Circle, fo here Prifms and Pyramids which ftand upon thofe Planes as their Bafes, be continually taken away from the Cones and Cylinders.

PROP. X. Theorem.

Fig. 20. *EVERY Cone is a third Part of a Cylinder, having the ſame Baſe and Heighth.*

Let a regular Polygon of as many Sides as you pleaſe be underſtood to be inſcrib'd in the Baſe C L, and upon it, as the Baſe, for a Cone let a Pyramid, and for a Cylinder, a Priſm be inſcrib'd. The Pyramid (*per* 7. *l.* 12.) will be a third Part of the Priſm. And if again in the Circle a Polygon of twice as many Sides be inſcrib'd, and upon it be inſcrib'd for a Cone a Pyramid, but for the Cylinder a Priſm; the Pyramid will again be a third Part of the Priſm. And thus it will always be. Wherefore ſeeing Pyramids end in a Cone, and Priſms in a Cylinder, the Cone alſo will be a third Part of the Cylinder. *Q. E. D.*

PROP. XI. Theorem.

Fig. 20, 21. *CONES of equal Heighth* (*BAF*, *QXR*) *are as their Baſes* (*CL*, *SE.*) *The ſame Thing belongs to Cylinders of equal Heighth alſo.*

Pyramids inſcrib'd into Cones of equal Heighth, are as their Baſes, (*per* 6. *l.* 12.) But Pyramids do at length end in Cones. Therefore Cones alſo are as their Baſes. And ſeeing Cylinders are three-fold of Cones, which have the ſame Baſe and Altitude with them, they alſo will be as their Baſes. *Q. E. D.*

Corollary.

IN the ſame manner it may be demonſtrated, that alſo Priſms and Cylinders of equal Heighth are betwixt themſelves as their Baſes; yea, that all cylindrical Bodies of the ſame Altitude; that is, which are produced from whatſoever Planes multiplied by the ſame Altitude, are betwixt themſelves as their Baſes. You may reaſon in the ſame manner of Pyramids and Cones of equal Altitude, and of all conical Bodies whatſoever.

PROP

PROP. XII. Theorem.

THE Proportion of like Cones (B A F and Q Z R) is triplicate to the Proportion of the Diameters (B F and Q R) which are in the Bases. The same thing is to be said of like Cylinders. Fig. 20, 21.

In the Bases of the like Cones, let regular Polygons be inscrib'd, which Polygons consequently will be like. The Pyramids which are inscrib'd upon these Polygons will also be like; as may be easily shew'd. Therefore their Proportion is triplicate (*per* 8. *l.* 12.) to the Proportion of the Sides B L, Q E; that is, to the Proportion of the Diameters B F, Q R. Wherefore seeing the Pyramids end in Cones, the Proportion also of the Cones is triplicate to the Proportion of the Diameters B F, Q R. *Q. E. D.*

The Theorem is manifest of Cylinders, seeing they are triple to Cones.

PROP. XIII. Theorem.

IF a Cylinder (B I) be cut with a Plane (R L,) parallel to the Bases (B Q, C I;) one Part, (B L,) shall be to the other Part, (R I,) as one Segment of the Axis (A O) is to the other Segment of the Axis (O F.) Fig. 22.

This Proposition is demonstrated, as at the first of *l.* 6.

The Theorem is in the same manner true of the Superficies.

PROP. XIV. Theorem.

CYLINDERS (A R and C I) of equal Bases (M Q, G H) are as their Altitudes (L Z, S F) The same thing happens to Cones. Fig. 23, 24.

Cut off from the higher Cylinder A R, the Cylinder A O, whose Heighth L E is the same with S F. Therefore (*per* 11. *l.* 12.) the Cylinders A O, C I, are equal. Seeing therefore the Cylinder A O, is to the Cylinder A R, (by

the foregoing) as L E is to L Z; C I alſo ſhall be to A R as L E is to L Z; that is, (becauſe L E and S F are equal by Conſtruction) as SF to LZ. *Q. E. D.*

Corollary.

THE Theorem is alſo true of Priſms, and likewiſe c Pyramids, and the Demonſtration together alike. Bu of Priſms, the thing is demonſtrated from *Corol.* 1. *p.* 9. 12. and 25. *l.* 11. and its *Corol.* Of Pyramids from this and from *p.* 7. *l.* 12.

PROP. XV. Theorem.

Fig. 24, 25. *EQUAL Cylinders (AR, D F) have their Baſe and Altitudes reciprocally proportional; an if they have them ſo, they are equal. The ſam thing is true of Cones.*

This is demonſtrated, as *Prop.* 34. *l.* 11. only for 32, an 25. *l.* 11. there cited, there muſt be cited here *Propoſiti* 11, and 13. *l.* 12.

Scholium.

Fig. 25, 24. WHEREAS *Euclid* hath ſaid nothing of compound Pr portion in Bodies, we ſhall briefly demonſtrate it this Place.

1. A Cylinder hath to a Cylinder, and a Priſm to Priſm, a Proportion compounded of the Proportions of t Baſes and Altitudes.

Let F D and A R be Cylinders of different Altitudes (f in thoſe of equal Altitude the Thing is manifeſt) Fro the higher, cut off AO of equal Heighth with F D. A let the Proportion be thus; as the Baſe U T is to the Ba M Q, ſo F N to X; and as the Altitude N D or B is to the Altitude B R, ſo is X to Z. We muſt therefo ſhew, that the Cylinder F D is to the Cylinder A R, FN is to Z. The Cylinder FD is to the Cylinder A (*per* 11. *l.* 12.) as the Baſe V T is to the Baſe M Q; th is, (by Conſtruction) as F N is to X; but the Cylinder A is to the Cylinder A R (*per* 13. *l.* 12.) as B O to B R, th

, (by Construction) as X to Z. Therefore, by Propor-on of Equality, the Cylinder F D is to the Cylinder A R, s F N to Z.

The Proposition may be demonstrated in the same manner f Prisms, but from *Cor.* 1. *p.* 9. and *Cor. p.* 14.

2. A Cone hath also to a Cone, and a Pyramid to a Py-amid, a Proportion which is compounded of the Propor-ons of Base to Base, and Altitude to Altitude:

For (by *Prop.* 10, and 7. *l.* 12.) they are third Parts of ylinders and Prisms.

PROP. XVI, XVII.

THESE Propositions, the most prolix of all other, have no other Use than to serve to the emonstrating, Prop. 18. *we shall demonstrate in nother more easy Way.*

Lemma to Proposition 18.

YLINDERS inscrib'd in an Hemisphere end in the Hemisphere. Let P Z Y be the greatest Semi-circle of e Hemisphere; and let the Radius A Z be perpendicular the Diameter PY. Cut A Z into a certain Number of ual Parts, A M, M N, N Z; and there being drawn rough the Points of the Divisions M, N, the perpendicu- Lines B O, &c. Let there be inscrib'd in the Semi-cles the Rectangles O B R K, E D H S; which after-ards being continued without the Semi-circle, let there be derstood to be describ'd about the Semi-circle the Rect-gles F T Y P, L V B O, Q X D E. They will all of em be of the same Heighth, and the Excesses of the cumscribed ones, above those which are inscrib'd, will be e Planes F K, L S, X E, V H, T R, which, taken toge-er, make the Rectangle F T Y P. For because X E is ual to D S, those L S, V H, X E, together, will be ual to the Rectangle L B that is, O R. Wherefore if u add on both Sides the Planes F K, T R, all those, F K, S, X E, V H, T R, taken together, will be equal to the ctangle F T Y P. If now the Semi-circle, with the Rect-gles, be understood to be turn'd about the Radius A Z, ich is in the mean while unmov'd, the inscribed Rect- *Fig.* 26.

angles E H, O R, will produce Cylinders inſcribed in th Hemiſphere; and the circumſcribed Rectangles will pro duce Cylinders circumſcribed about the Hemiſphere, ſtand ing one upon another; and as the Exceſſes of the circum ſcribed Rectangles above the inſcribed ones, was the Rect angle F Y; ſo likewiſe the Exceſſes of the circumſcribe Cylinders above the inſcribed ones, will be the Cylinde which is produced from the Rectangle F Y. But now th Altitude of this Cylinder will be made leſs than any give Heighth; and conſequently (as is manifeſt from 13. *l.* 12. it ſelf will grow to be leſs than any given Cylinder, if th Radius being divided into more equal Parts without En the Number of Rectangles, and from thence of Cylinder be infinitely encreaſed. Therefore the Exceſs of the ci cumſcribed Cylinders, and much more of the Hemiſphe it ſelf, which is only a Part of the circumſcribed Cylinde above the inſcribed ones, will at laſt become leſs than ar given one. Therefore (by *Defin.* 6. *l.* 12.) Cylinders in nitely inſcribed in an Hemiſphere, do at length end in t Hemiſphere itſelf. *Q. E. D.*

Corollary.

IN the ſame manner it will be demonſtrated, that Cyli ders inſcribed in a Cone, Conoid, Spheroid, *&c.* do laſt end in the ſame.

PROP. XVIII. Theorem.

Fig. 27. *THE Proportion of Spheres is triplicate to t Proportion of their Diameters* (*B K, R Z.*)

The Radius's A B, Y R, being divided into as many eq Parts as you will, but of an equal Number, and there l ing drawn through the Points of the Diviſions perpendi lar, *&c.* Let Rectangles of an equal Number be und ſtood to be inſcribed in the greater Semi circles of Spheres, which Rectangle bring turned about, the unmo Radius's A B, Y R, will be conceiv'd to inſcribe in both Hemiſpheres, a like Number of Cylinders ſtanding one on another. Now, becauſe K C is (*per Cor. p.* 3. *l.* 6.) C F, as C F is to C B; the Proportion of K C to C B (

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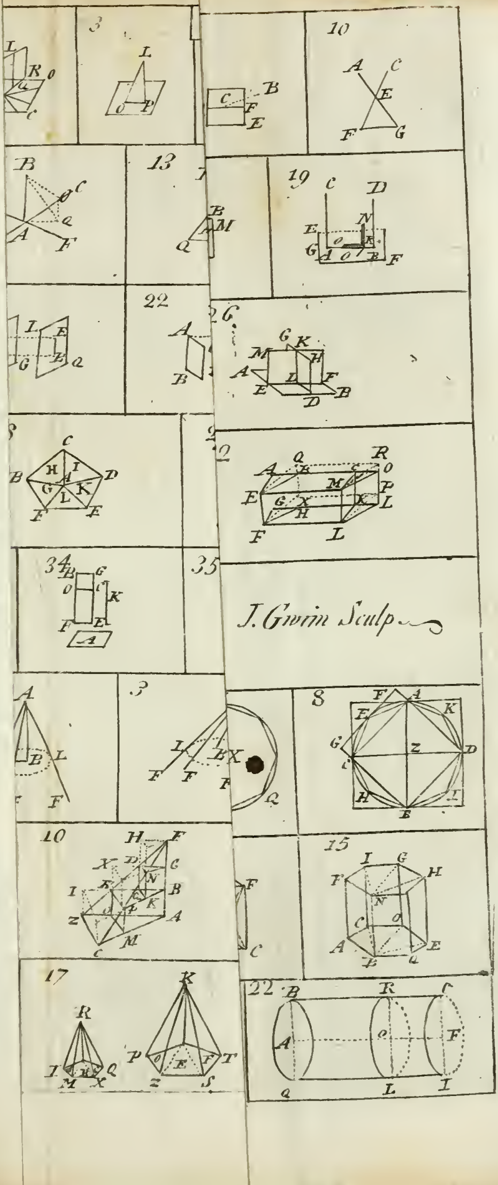
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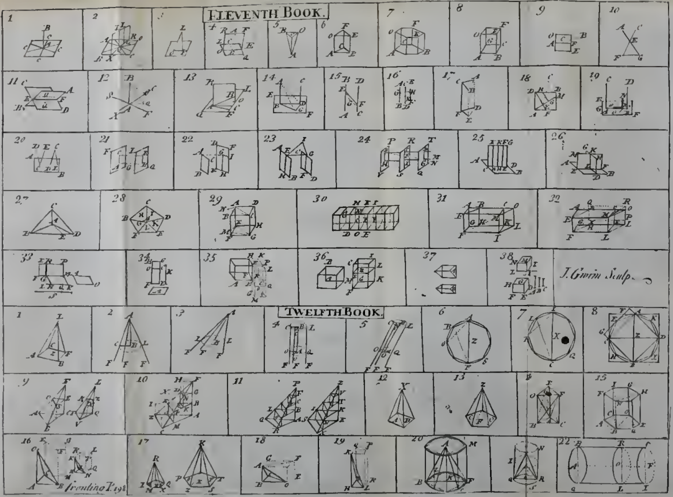

ELEVENTH BOOK.
TWELFTH BOOK.
J. Gwin Sculp.
fronting P. 198

Defin. 10. *l.* 5.) will be duplicate to that of K C to C F, that is, to the Proportion of F C to C B. In like manner the Proportion of Z E to E R, will be duplicate to the Proportion of X E to E R. But, by the Conſtruction, K C is to C B, as Z E is to E R. Therefore F C alſo is to B C, as X E to E R. But B C, by the Conſtruction, is to C O, as R E to E S. Therefore by Equality, F C is to C O, as X E is to E S. Therefore (by *Defin.* 4. *l.* 12.) the Cylinders F L, X Q, are like, and conſequently their Proportion is (*per* 12. *l.* 12.) triplicate to the Proportion of their Diameters, F I, X V, or of the Semi-diameters, F C, X E, which are the Baſes. But the Proportion of F C to X E, is the ſame with the Proportion which is betwixt the Diameters of the Spheres B K, R Z; for, as I have already ſhew'd, F C is to X E, as C O is to E S; that is, as B K is to R Z, which, by the Conſtruction, are Equi-multiples of thoſe, C O, E S.) Therefore the Proportion of the Cylinders, F L, X Q, is triplicate to the Proportion of the Diameters, B K, R Z. In the ſame manner we might demonſtrate that each Cylinder inſcribed in one Hemiſphere, bears to each Cylinder inſcribed in the other Hemiſphere, a Proportion triplicate to the Proportion of the Diameters, B K, R Z. Therefore alſo the Proportion of all together, to all together, is triplicate to the Proportion of the Diameters B K, R Z. Wherefore ſeeing the Aggregates of the Cylinders do at length end in their Hemiſpheres, the Proportion of the Hemiſpheres alſo, and conſequently of the Spheres, will be triplicate to the Proportion of their Diameters. *Q. E. D.*

Corollary.

THEREFORE the Proportion of the Diameters being known, the Proportion of the Spheres becomes known likewiſe. As if the Diameter of the leſſer be one Foot, that of the greater ten Feet; let the Proportion of one to ten be continued through four Terms, 1, 10, 100, 1000; as 1, the firſt is to 1000, the fourth Term, ſo is the leſſer Sphere to the greater.

The Dimenſion of Cones, Cylinders and of the Sphere, will be exhibited in the following Book out of *Archimedes*.

Scholium.

Scholium.

AS like plain Figures are encreas'd or diminiſh'd in any given Proportion by one mean Proportional, ſo like Bodies are encreaſed or diminiſhed by two mean Proportionals.

Let a Sphere, or Cube, or any other Body whatſoever, be given, whoſe Radius, or Side, is A. Likewiſe let any Proportion whatſoever of A to B be given, as the double, or 2 to 1. A Body is to be diſcovered both double to the given one, and like to it.

Betwixt the Terms of the given Proportion A and B, let there be found two mean Proportionals X, Z, according to what was taught in the *Scholium* of *Prop.* 13 *l. 6.* A Sphere, whoſe Radius is X, or other Body like to the given one, which is made upon the Side X, will be double to the given one.

For like Bodies, whoſe Radius's, or Sides, are A and X, have betwixt themſelves a Proportion which is triplicate to the Proportion of A to X, (by *Corollary*, *Propoſition* 9. and by *Propoſition* 12. and 18. *L.* 12.) that is, the ſame (*per Defin.* 10. *l.* 5.) which A hath to B.

And this is that moſt celebrated Problem which, from *Apollo* and *Delos*, is called the *Deliacal* Problem; becauſe at the Time of a moſt grievous Peſtilence, which waſted *Athens*, being conſulted, he gave Anſwer, that the Peſtilence would ceaſe, if his Altar, which was of a cubical Form, were doubled. Thus *Valerius Maximus*, *L.* 8.

THEOREMS

SELECTED OUT OF

ARCHIMEDES:

By ANDREW TACQUET,

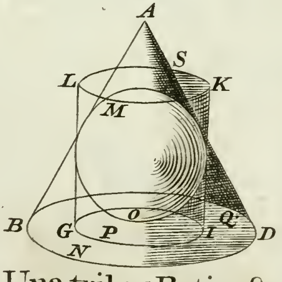
OF THE

SOCIETY of JESUS;

And Demonſtrated in a more Eaſy and Compendious Way.

To which are added,

Some other agreeable Propoſitions, newly invented, by the ſame ANDREW TACQUET.



Una tribus Ratio eſt.

DUBLIN:

Printed in the Year MDCCLIII.

TO THE

READER.

ALBEIT *there have appear'd very many most excellent and admirable Men in the Mathematical Sciences; yet the chief Glory of all hath always, by a certain common Consent, been given to* Archimedes *of* Syracuse. *Tho' indeed, more there are who commend, than who read him; more who admire, than understand him. The Causes of which Neglect seem to be these, the Bulk and Scarceness of Copies, some Obscurity of the Translation, which is directly made out of the* Greek *Language, together with the Prolixity and Difficulty of his Demonstrations. I judged therefore that it would be for the Profit of studious Learners, if after my Illustration of the Elements, I should subjoin these* Theorems *which had been selected by me out of* Archimedes, *and demonstrated in a much easier and briefer Way. Furthermore, I have selected those, which bring along with them both more of Admiration and of Benefit; and have in my Demonstration took such a Method, that, I hope, he who understands the Elements, will, without*

any

any great Labour, comprehend these most excellent Inventions of the Prince of Geometricians. I have also added at the End, thirteen Propositions, and thereby enlarged the Doctrine of Archimedes *concerning the Sphere and Cylinder: Where, amongst other Things, I demonstrate, that the Sesquialteral Proportion is continued in the Three Bodies, a Sphere, a Cylinder and Equilateral Cone; both the latter being inscrib'd about the Sphere. Moreover, I have added divers Things here and there, amongst which the* 12*th* Proposition, *and the* Corollaries *of* Proposition 14. *are the chief; and several* Scholiums. *Make use of these Discoveries whosoever thou be'st, that art a Candidate of* Geometry; *and how much thou hast improv'd in* Euclid, *make Proof of in* Archimedes. *And when thou perceivest thy self to be fix'd and rais'd upwards in the Contemplation of the most noble Truths, raise up thy Mind, while it is thus already lifted up from these lower Things, yet higher, and direct it to that Truth, which is Original, Eternal, Immense, and is no other than* GOD; *by the ineffable Vision of whom, I trust we shall hereafter be made eternally Happy. Farewel.*

THEO-

THEOREMS

Selected out of ARCHIMEDES.

DEFINITIONS.

Or an Explanation of certain Terms.

LET there be a Circle B E C G, whose Center is A, its Diameter B C, which let the right Line E G cut at right Angles, (but not through the Center) in D. Let there be drawn from the Center the Radius's A E, A G. This being suppos'd, *Fig. 23. Of the Table out of Archimedes.*

NOTE, (1.) That a Sector of a Sphere is that which is produced from the Sector of the Circle A E C G, or A E B G, turn'd round about the Diameter B C.

2. That a Segment, or Portion of a Sphere, is that Part of it which is produced from the Segment of the Circle E C G or E B G turn'd round about the same Diameter B C.

3. The Vertex, or Top of the Spherical Portion E B G, is the Extremity B of the unmov'd Diameter; the Basis, the Circle describ'd by E G; the Axis, that Part of the Diameter B D, which is intercepted betwixt the Top B, and D the Center of the Base.

4. When I name the Superficies of a Spherical Portion, or of a Body inscrib'd in it, or of a Cone, I always understand it without the Base; and when I say the Superficies of a Cylinder, I mean likewise without the Bases; unless the Word [*whole*] be adjoin'd to [Superficies;] for then the Bases also are to be taken in.

Again, when I treat of Cylinders or Cones, I speak of no other than right ones.

Axioms

Axioms.

Fig. 1, 16. 1. THE Circuit of a Polygon infcrib'd in a Circle is lefs than the Circumference of the Circle.

Fig. 1. 2. The Circuit of a Polygon defcrib'd about a Circle is greater than the Circumference of the Circle.

Fig. 16. 3. And if a Polygon infcrib'd in a Circle, be turn'd about the Diameter (A E) together with the Circle, the Superficies of the Body produced by the Polygon, will be lefs than the Superficies of the Sphere. And if a Polygon circumfcrib'd about the Circle, be turn'd about the Diameter, together with the Circle, the Superficies of the Body produced by the Polygon, will be greater than the Superficies of the Sphere.

Fig. 17. 4. In like manner the Circuit of a Polygon infcrib'd in a Segment of a Circle (D A F) is lefs than the Circumference of the Segment. And if a Polygon infcrib'd in the Segment, be together with the Segment (A O) turned round; the Superficies of the Body produced by the Polygon will be lefs than the Superficies of the Spherical Portion D A F.

Fig. 3, 6. 5, The Superficies of a Prifm infcrib'd in a Cylinder, is lefs than the Superficies of the Cylinder; but the Superficies of the Prifm which is circumfcrib'd is greater.

Fig. 4, 8. 6. And the Superficies of a Pyramid infcrib'd in a Cone, is lefs than the Superficies of the Cone; but the Superficies of a circumfcrib'd Pyramid is greater.

PROPOSITIONS I, II.

ARE not neceffary.

PROP. III. Theorem.

THE Circuits or Polygons circumfcrib'd about and infcrib'd in a Circle, do at laft end in the Circumference of the Circle. In like manner the Polygons themfelves do at laft end in the Circle.

Fig. 1. *Tabl. Archimed.* If, to wit, the Arches being bifected without End, more and more Sides be circumfcrib'd about and infcrib'd in the Circle.

Part

Part I. Let there be underſtood to be inſcrib'd in and deſcrib'd about a Circle, regular Polygons; whether it be done ſo as is ſet down, *Prop.* 12. *l.* 4. or as in the preſent Figure, the Thing will be the ſame. It is manifeſt (*per Corol.* 1. *p.* 4. *l.* 6.) that F I is to C E (that is, the whole Circuit circumſcrib'd, unto the whole Circuit inſcrib'd) as I A is to C A. But I C, the Exceſs of the right Lines I A above C A, becomes at length leſs than any given Line, if more and more Sides be underſtood to be infinitely circumſcrib'd and inſcrib'd; therefore alſo the Exceſs of the Circuit circumſcrib'd above that which is inſcrib'd, will at length become leſs than any given Line. Therefore much more the Exceſs of the Circuit circumſcrib'd above the Circumference of the Circle, will be leſs than any given one. In like manner, becauſe I have already ſhew'd the Defect of the Circuit inſcrib'd, whereby it falls ſhort of that which is circumſcrib'd, to be leſs than any given Line: Therefore much more will the Defect of the Circuit inſcribed, whereby it falls ſhort of the Circumference of that Circle, become leſs than any given Line. The Circuits therefore, as well as that which is inſcrib'd, as well as that which is circumſcrib'd, do at length (*Def.* 6. *l.* 12.) end in the Circumference. Which was the firſt Part. To demonſtrate theſe Things further is not worth the while, ſeeing they are manifeſt enough.

Part II. Becauſe it hath already been ſhew'd that the Exceſs F I, above the Side E C, becomes at length leſs than any given Line (F I is to E C, as I A to C A;) therefore alſo the Exceſs of the Square of F I, above the Square of E C, will become at length leſs than any given one. But as the Square of F I, is to the Square of E C, ſo (*per* 20. *l.* 6.) is the Polygon circumſcrib'd, to that which is inſcrib'd. Therefore the Exceſs of the Polygon circumſcrib'd above that which is inſcrib'd, will alſo become at length leſs than any given one. Therefore much more will the Exceſs of the Polygon circumſcrib'd above the Circle, become at laſt leſs than any given one; and conſequently, the Defect alſo of the Polygon inſcrib'd, whereby it falls ſhort of the Circle, will at length become leſs than any given Defect. Therefore Polygons, as well inſcrib'd as circumſcrib'd, do at laſt (*Defin.* 6. *l.* 12.) end in the Circle. Which was the ſecond Part.

PROP.

PROP. IV. Theorem.

(c) *Per Defin.* 3. *l.* 4. *Fig.* 1.

A Regular (c) Polygon (F I N T R) circumscrib'd about a Circle, is equal to a Triangle whose Base is the Circuit of the Polygon, and its Heighth the Radius of the Circle.

And a regular Polygon inscrib'd in a Circle is equal to a Triangle, which hath for its Base the Circuit of the Polygon, and for its Heighth the Perpendicular (A O) let down into one Side from the Center.

Part I. The Radius A B, drawn to the Point of Contact, is (*per* 18. *l.* 3.) perpendicular to the Tangent I F. Wherefore, if the right Lines A F, A I, A N, *&c.* being drawn, the Polygon be resolv'd into Triangles; the Radius A B will be the common Altitude of all; and consequently it is manifest that the Triangles are equal. Therefore a Triangle, which hath its Base equal to the Sum of the Sides F I, I N, N T, *&c.* and A B for its Altitude, will (as is manifest from 1. *l.* 6.) be equal to them all, that is, to the whole Polygon circumscrib'd.

Part II. This may be concluded by the same reasoning as the other.

PROP. V. Theorem.

Fig. 2.

A Circle is equal to a Triangle, which hath for its Base the Circumference, and for its Heighth the Semi-diameter of the Circle.

Regular Polygons circumscrib'd about a Circle, and Triangles which have for their Bases the Circuit of the Polygon, and for their Altitude the Radius of the Circle, are always (by the foregoing *Proposition*) equal. But Polygons circumscrib'd infinitely about the Circle end in the Circle (by the third of this Book;) and in like manner Triangles (as I will shew by and by) which have for their Base the Circuit of the circumscrib'd Polygon, and for their Altitude the Radius A B, at last end in a Triangle, which hath the

Circum-

Circumference for its Bafe, and for its Altitude the Radius A B. Therefore (by the firft) a Circle and a Triangle, which hath the Circumference for its Bafe, and the Radius for its Altitude are equal.

But that Triangles contain'd under the Circuit of the Polygon, and the Radius of the Circle, end at laft in a Triangle, which is contain'd under the Circumference and the Radius, I thus fhew. Triangles under the Circuit of the circumfcrib'd Polygon, and the Radius A B, are to the Triangle, which is under the Circumference and the Radius AB, (by 1. *l.* 6) as Bafe to Bafe, that is, as the Circuit of the Polygon to the Circumference; fince this Triangle and the other have a common Altitude. But the Circuit of the Polygon (by the third) ends in the Circumference. Therefore the other Triangles end in this.

Corollaries.

1. FROM this and 41. *l.* 1. it is manifeft that a Rectangle under the Radius, and half the Circumference, is equal to the Circle; that one under the Radius, and the whole Circumference, is double; that one under the whole Circumference and the whole Diameter is quadruple thereto.

2. A Circle is to an infcrib'd Square, as half the Circumference (CDE) is to the Diameter; but to a Square circumfcrib'd, as the fourth Part of the Circumference is to the Diameter. *Fig.* 5. *l.* 4.

For the Rectangle under C D E, and the Radius C A or C F, that is, (by the foregoing *Corollary*) the whole Circle is to the Rectangle G F C E, to wit, the Rectangle under F G and C F (that is, to the infcrib'd Square B C D E) as (*per* 1. *l.* 6.) C D E, half the Circumference is to F G or C E the Diameter; which was the firft Thing. And confequently the Circle is to the double the Rectangle G F C E, (that is, to F H, the circumfcrib'd Square) as C D E is to the double of the Diameter C E, or as the Quadrant C D is to the Diameter C E.

[3. " Of Figures which are of equal Circumferences, " the Circle is the moft capacious. Let the Circumference " of any Polygon whatfoever (as for Inftance of a Square) " E G H I be equal to the Circumference of the Circle. I " fay, that the Area of the Circle is greater than that of " the Polygon. For the Area of the Circle is equal to a *Fig.* 30.

" Triangle, whose Base is the Circumference, and its Al-
" titude the Semi-diameter F A : And the Area of the Po-
" lygon is equal to a Triangle, whose Base is the Compass
" of the Polygon ; which, by the Hypothesis, is equal to
" the Circumference of the Circle, and which hath for its
" Altitude the Perpendicular F O, let down from the Cen-
" ter of the Circle unto the Side of the Polygon, which,
" since it is always less than the Radius of the Circle, it is
" manifest that the Area of the Polygon is less than the
" Area of the Circle. *Q. E. D.* And in like manner,
" amongst all solid Figures contain'd under equal Surfaces,
" the Sphere may be demonstrated to be the most capacious.]

PROP. VI. Theorem.

THE Circumference of a Circle contains the Diameter less than thrice and one seventh (or $\frac{10}{70}$;) and more than thrice and $\frac{10}{71}$.

For the Demonstration of this Theorem, *Archimedes* assumes regular Polygons, one circumscribed about a Circle, the other inscribed, and both of them of 96 Sides. And then he shews that the 96 inscribed about a Circle, do contain the Diameter less than thrice and one seventh, and consequently that the Circumference which is less than them, doth also contain the Diameter less than thrice and one seventh. But the 96 Sides inscribed in the Circumference, (and consequently the Circumference also which is greater than them) doth contain the Diameter more than three times $\frac{1}{7}$. But this Demonstration is too long to be brought in this Place. Nay, if we minded to extend our Geometrical Reasoning to Polygons of more Sides still, we may contract the Limits even now set more and more without Limit, and so come nearer and nearer for ever to the true Proportion. This hath been perform'd by *Ludolph à Ceulen Grimberger*, *Metius*, *Snellius* and others. The chief Proportions hitherto found I shall here subjoin.

[" However, since a Tangent of 30 Degrees, multiplie[d]
" by 12, gives the Circuit of a circumscribed Hexagon
" and a Sine of 30 Degrees multiplied by 12, gives th[e]
" Circuit of an Hexagon, which is inscribed : Forasmuc[h]
" also as in like manner the Tangent of half a Degree mul-
" tiplie[d]

" tiplied by 720, yields the Circuit of a circumſcribed " Polygon of 360 Sides; and the Sine of half a Degree, " the Circuit of an inſcribed Polygon of 360 Sides; and " ſo on for ever: It will not be difficult to underſtand, by " what Means many ſuch Numbers may be found, out of " the now given Tables of Sines and Tangents.]

The firſt Proportion, which is that of *Archimedes*, is thus:

The Diameter 7.
The Circumf. is 22; which is greater than the true.
The Diameter 71.
The Circumf. is 223; leſs than the true one.

The Proportion of 22 to 7, and 223 to 71, if they be reduced to a common Conſequent, (which is done after the ſame manner, in which Fractions are reduced to the ſame Denomination) will be thus, 1562 to 497, and 1561 to 497.

Therefore the Diameter being ſuppos'd 497 Parts, the Circumference, greater than the true one, will be 1562; and the Circumference leſs than the true, 1561.

Both of them therefore differ from the true, by a Quantity leſs than $\frac{1}{497}$ Part of the Diameter. But if the Proportion of 7 to 22, and 71 to 223, be reduced to a common Conſequent, there will ariſe the Proportions of 1561 to 4906, and of 1562 to 4906.

Therefore the Circumference being ſuppos'd to be 4906 Parts, the Diameter leſs than the true, will be 1561, the Diameter greater than the true, 1562.

Both therefore differ from the true Diameter by Quantity leſs than $\frac{1}{4906}$ Part of the Circumference.

The Proportion delivered by *Metius* is much more accurate than this of *Archimedes*. According to this,

The Diameter is 113.
The Circumference 355.

Amongſt all Proportions conſiſting of ſmall Numbers, none comes nearer to the true one; for from this, the Diameter being ſuppos'd of 10,000,000 Parts, the Circumference comes to be of 31,415,929, which differs from the true one only in the firſt Figure 9, and this by an Exceſs, but a little greater than two ten-millioneth Parts of the Diameter.

But more exact than both, is that double Proportion of *Ludolphus a Ceulen*; the former of which consists of 21 Figures, and the latter of 36.

The Diameter.
100,000,000,000,000,000,000.
The Circumf. greater than the true.
314,159265,358979,323847.
The Circumf. less than the true,
314,159265,358979,323846.

The Difference of both the Circumferences is one Particle of the Diameter denominated from a Number which consists of a Unity and 20 Cyphers; and consequently, as well this as that, differs from the true Circumference by a Quantity less than is the said small Part of the Diameter; to wit, one hundredth of a millioneth of a millioneth of a millioneth Part.

The Diameter,
100000,000000,000000,000000,000000,000000.
The Circumf. greater than the true,
314159,265358,979323,846264,338327,950289.
The Circumf. less than the true,
314159,265358,979323,846264,338327,950288.

The Difference of both the Circumferences, betwix[t] which is the true one, is that small Part of the Diameter denominated from a Number which consists of Unity an[d] 35 Cyphers; which small Part bears a less Proportion t[o] the whole Diameter, than one little Grain of Sand doth t[o] the whole Globe of the Earth. For the whole Globe o[f] the Earth doth not consist of so many little Grains of Sand as are the little Parts of the said Sort which are contain'd i[n] the Diameter.

It is needless to go any further. Nevertheless you ma[y] proceed infinitely, if you be minded to continue Geometrical Reasoning, an expedite Method of which is delivered b[y] *Snellius.*

[*The Circumference being suppos'd of*
1,000000,000000,000000,000000,000000,000000 *Parts*
The Diameter will be as near as may be, of
0,318309,886183,790671,537767,926745,028724 *Parts*

Scholium

Scholium.

THE most excellent Advantages of the Proportion now delivered, are these which follow.

The Invention of the Diameter from the Circumference.

SET the greater Term of one of the Proportions which have been now delivered in the first Place, the lesser in the second, the Circumference in the third; by these three Numbers let there be sought by the Golden Rule a fourth Proportional. That is the Diameter sought.

As if the Circumference of the greatest Circle of the Earth be suppos'd to contain 25000 *English* Miles of 5280 Feet each, and the Diameter be sought; the Terms will stand thus,

355—113—25000—7854.

Multiply now the second by the third, and divide the Product by the first; and there will arise 7854 Miles for the Diameter of the Globe of the Earth.

The finding out of the Circumference from the Diameter.

LET the lesser Term of one of the Proportions above delivered be set in the first Place; the greater in the second; the known Diameter in the third: And by these three Numbers, let there be sought a fourth Proportional. That will give the sought Circumference.

As if the Diameter of the Globe of the Earth be suppos'd to contain 7854 *English* Miles; and the Circuit is sought; the Terms will stand thus,

113——355——7854——25000.

Then multiply the second by the third, and divide the Product by the first; there will arise 25000 Miles for the Circumference of the Globe of the Earth.

How little this Circumference exceeds the true one, was said above; to wit, by an Excess but a little greater than are two ten-millioneth Particles of the Earth's Diameter; that is, by 9 or 10 Feet. But if we use the *Ludolphin* Proportion, even the former, the Terms whereof consist

of 21 Figures; there will be found a Circumference insensibly differing from the true, not only when the given Diameter is of 7854 Miles, such as is the Diameter of the Earth; but also although the Diameter be suppos'd of an 100 Millions of those Miles. For this being suppos'd, there will arise a Circumference differing from the true one by a Quantity about one hundred millioneth Part of a Foot. But if to find out the Circumference of the Globe of the Earth, we make use of the Proportion of *Archimedes*, the Interval of the two Circumferences, the one greater, the other less than the true one, will exceed 20 Miles. *Archimedes* his Proportion therefore is not to be used but in small Measures; nay, it will always be expedient to use that of *Metius*, which both consists of small Terms, and is above a 1000 Times more exact.

The measuring of a Circle.

THE Semi diameter multiplied by half the Circumference, produceth the Area of the Circle; as is manifest from *Corollary* 1. *Proposition* 5 of this Book.

As if the Semi-diameter of the Earth, which contains 3927 Miles, be multiplied by half its Circumference, to wit, by 12500, there will arise 49,075500 Miles Square for the Area of the greatest Circle of the Earth. The Difference of the circular Area thus found from the true is had, if the Difference of half this found Circumference from the true half Circumference be multiplied by the given Semi-diameter; or the Difference of this Semi-diameter from the true, be multiplied by the given Semi-circumference.

The Mensuration of Cylinders and Cones.

I put this here, because it depends upon the Mensuration of a Circle. A Cylinder therefore, and any Prism whatsoever, is produced from the Altitude multiplied by the Base: A Cone and Pyramid, from the third Part of the Altitude, multiplied by the Base; for they are third Parts of Cylinders and Prisms, having the same Base and Altitude with them, by 10, and 7. *l.* 12.

Let the Base of a Cylinder or Cone, be of 50 Square Feet, and the Height of 100 Feet. Multiply 100 by 50, and there arise 5000 cubic Feet for the Solidity of the Cylinder

der. Multiply the third Part of the Altitude 100, which is $33\frac{1}{3}$ by 50, there arise $1666\frac{2}{3}$ cubical Feet for the Solidity of the Cone.

PROP. VII. Theorem.

THE Circumferences of Circles have the same Proportion betwixt themselves which their Diameters have. Fig. 6, 7. l. 12.

For the Circuits of like Polygons, which may be inscribed in a Circle without End, are always betwixt themselves, as the Diameters A F and I C (by *Corollary, p.* 1. *l.* 12.) But these Circuits (by the 3d *Proposition* of this Book) end at length in the Circumference. Therefore their Circumferences also are betwixt themselves as their Diameters. *Q. E. D.*

PROP. VIII. Theorem.

THE Superficies of a Prism, as well that which is circumscrib'd about, as that which is inscrib'd in a Cylinder, is equal to a Rectangle, whose Heighth is the Side of the Cylinder, but its Base equal to the Circuit of the Base of the Prism.

Part I. The Superficies of the circumscribed Prism touches the Cylinder according to the Lines E A, N F, &c. which are the Sides of the Cylinder; but these (because by the Hypothesis the Cylinder is a right one) are right to the Plane of the Base, and consequently right also (by *Definition* 3. *L.* 11.) to the Lines C G, G M, &c. But they are also equal betwixt themselves. Therefore one Side of the Cylinder is the common Heighth of all the Rectangles C O, O M, M H, &c. Therefore the Superficies of the circumscribed Prism is equal (as is manifest from 1. *L.* 6.) to a Rectangle contain'd under the Circuit of the Base of the Prism, and the Side of the Prism or Cylinder. Fig. 3.

Part II. The Reason of this is the same. For the Side of the Cylinder is again the common Altitude of the Rectangles B D I K, K I P Q, &c. which constitute the Superficies of the inscribed Prism.

PROP.

PROP. IX. Theorem.

Fig. 4. *THE Superficies of a regular Pyramid circumscrib'd about a right Cone, is equal to a Triangle, which hath for its Base the Circumference (F H L D) of the pyramidal Base, but its Heighth the Side of the Cone (B G.)*

And the Superficies of a regular Pyramid inscribed in a right Cone, is equal to a Triangle, which hath for its Base the Circumference of the pyramidal Base, but for its Heighth the Perpendicular (B O) let down from the Top unto a Side of the Base.

Part I. Let there be drawn unto the Contacts G, K, M, the right Lines B G, B K, B M. These will all be Sides of a right Cone, and consequently equal. And, because (by the Hypothesis) the Axis B A is perpendicular to the Plane of the Base F K D, the Plane also G B A (*per* 18. *l.* 11) will be perpendicular to the Plane F K D. But H G (*per* 18. *l.* 3.) is perpendicular to A G, the common Section of the Planes F K D and G B A. Therefore H G (as is gathered from *Defin.* 4. *l.* 11.) is also perpendicular to the Plane G B A. And consequently is also perpendicular to B G. Therefore the Side G B of the Cone, is the Heighth of the Triangle F B H. In the same manner the Side of the Cone will be the Heighth of the rest, H B L, L B D. *&c.* Therefore the Triangle comprehended under the Circumference F H L D, and the Side of the Cone is equal to the Superficies of a Pyramid circumscribed, without the Base. Which was the first Part.

II. The Demonstration of this Part is almost the same with that of the former.

PROP. X. Theorem.

THE Superficies of a regular Prism circumscrib'd about a right Cylinder, ends (Def. 6. l. 12.) *in the Superficies of the Cylinder; and the Superficies*

Superficies of a Pyramid circumſcrib'd about a right Cone ends in the Superficies of the Cone.

Part I. The Superficies of regular Priſms deſcribed *Fig.* 3. about and inſcribed in a Cylinder without end will have at laſt a Difference betwixt themſelves leſs than any which can be given. Much more therefore will the Superficies of a circumſcribed Priſm differ from the Superficies of the Cylinder, which is in the middle between the inſcribed and circumſcribed Superficies, by a Difference leſs than any given one whatſoever; that is, (*Def.* 6. *l.* 12.) will end in the cylindrical Superficies, whilſt it continually exceeds it leſs and leſs.

Part II. This may be ſhewed in the ſame manner from *Fig.* 4. the 9th and 3d of this.

In the Figures there are only exhibited the Halves of the Cylinder and Cone, leſt a Multitude of Lines ſhould breed Confuſion. But the Cylinder and Cone are to be conceiv'd in the Mind entire, and as having theſe circumſcribed Priſms and Pyramids encompaſſing them. For thus it more clearly appears that plain Surfaces circumſcribed are greater, according to the third *Axiom.*

A Lemma to the following Propoſition.

LET AB, CD, EF, be proportional, and let KB be *Fig.* 7. half AB, and EG, double EF; KB, CD, EG, will alſo be proportional.

The right Line KB is to AB, as EF is to EG. Therefore the Rectangle KB, EG (*per* 16. *l.* 6.) is equal to the Rectangle AB, EF. But this (by 17. *l.* 6.) is equal to the Square of CD. Therefore alſo the Rectangle KB, EG is equal to the Square of CD. Therefore (by 17. *l.* 6.) KB, CD, EG are proportional.

PROP. XI. Theorem.

A Circle, whoſe Radius (GH) is a mean Proportional betwixt the Side of a right Cylinder (BC) and the Diameter of the Baſe (BD) is equal to the cylindrical Superficies. *Fig.* 5, 6.

Let the regular, and conſequently like Polygons, NM, RS, be underſtood to be circumſcribed about the Circles ABN, GPH; and upon the Polygon NM, let a Priſm be

be conceiv'd to be erected, with which the Cylinder is circumscribed. Because B D, G H, B C are, by the Hypothesis, proportional, A D also, (or A N) G H, and the double of B C will, by the *Lemma*, be proportional. Now the Triangle contain'd under A N, and the Circuit of the Polygon M N is equal to the Polygon circumscribed N M (by the fourth of this Book.) And the Rectangle under B C, or E F, and the same Circuit N M, (that is, as is manifest from 41. *L.* 1. the Triangle under the Circuit N M, and the double of B C) is equal (by the eighth of this Book) to the Superficies of a Prism circumscrib'd about the Cylinder. But a Triangle under the Circuit N M and A N, is to the Triangle under the Circuit N M, and the double of B C (by 1. *l.* 6.) as A N is to the double of B C. Therefore the Polygon N M also is to the Superficies of a Prism circumscrib'd about a Cylinder, as A N is to the double of B C. But because I have already shewed A N, G H, and the double of B C to be proportional, the Proportion of A N to the double of B C is (by *Def.* 10 *l.* 5.) duplicate to the Proportion of A N to G H. Therefore the Polygon N M hath to the Superficies of the Prism a Proportion duplicate to the Proportion of A N to G H. But the Polygon N M hath also to the Polygon like to it, G R Q S, a Proportion duplicate to the Proportion of A N to G H, as is easily gathered out of 1. *l.* 12. Therefore the Polygon N M hath the same Proportion to the Superficies of the Prism, which it hath to the Polygon G R Q S; which consequently is equal to the Superficies of the Prism. In the same manner I might shew, that the prismatic Superficies, which are circumscriptible infinitely about the Cylinder, are always equal to the Polygons which may be circumscribed infinitely about the Circle G P H. Wherefore seeing both the prismatic Superficies (by the 10th of this) end in the Surface of the Cylinder, and the Polygons in the Circle G P H (by the 3d of this) the Superficies of the Cylinder also will be equal to the Circle G P H. *Q. E. D.*

From this admirable Theorem, a Circle is presented, which is equal to a cylindrical Superficies.

Corollaries.

Corollaries.

THE Superficies of a right Cylinder is equal to a Rectangle contained under the Side (B C) and the Circumference of the Base. *Fig. 5, 6.*

The double of B C (as hath been shew'd above) is to G H, as G H is to B A, or AN ; that is, (by the 7th of this) as the Circumference P is to the Circumference B N. Therefore the Triangle under the first, to wit, the double of B C ; and the fourth, to wit, the Circumference B N, is equal to a Triangle under the second G H, and the third, to wit, the Circumference P, (as appears from 16. *l.* 6. But the Triangle under G H, and the Circumference P, is (by the 5th of this) equal to the Circle G P H, that is, by the 11th of this) to the cylindrical Superficies. Therefore also the Triangle under the double of B C, and the Circumference B N, (that is, as appears from 41 *l.* 1. the Rectangle which is under B C and the Circumference B N) will be equal to the cylindrical Superficies. *Q. E. D.*

From this *Corollary* it is manifest, that the Properties of Rectangles are common to them with cylindrical Superficies. Therefore let this be *Corollary* 2.

2. The cylindrical Superficies (B M, Q N) which are of the same Height, are betwixt themselves as the Diameters of their Bases (B F, Q R.) *Fig.* 20, 21. *l.* 12.

For the Rectangles under the Circumference (C L, S E and the equal right Lines F M, R N, to which (by *Coroll.* 1) the cylindrical Superficies are equal, are betwixt themselves (by 1. *l.* 6.) as the Bases, to wit, the Circumferences C L, S E ; that is, as the Diameters B F, Q R, (by the 7th of this)

3. The cylindrical Superficies (C I, A R) which have equal Bases, are betwixt themselves, as their Altitudes (T I, B R.)

For the Rectangles contain'd under the equal Circumferences G B, M Q, and the Sides T I, B R, to which (by *Coroll.* 1.) the cylindrical Surfaces are equal, are betwixt themselves (by 1. *l.* 6. as T I, B R. *Fig.* 23, 24. *l.* 12.

4. Like cylindrical Surfaces (B M, R I) have betwixt themselves a Proportion duplicate to that which (B F, Q R) the Diameters of the Bases have. *Fig.* 20, 21. *l.* 12.

Seeing

Seeing the Cylinders are ſuppos'd to be like, M F will be to I Q (by *Defin.* 4. *l.* 12.) as B F is to Q R; that is, (by the 7th of this) as the Circumference C L to the Circumference SE. Wherefore the Rectangles alſo which are contain'd under the Circumferences C L, S E, and the Side M F, I Q, will be like; and conſequently they will have betwixt themſelves (by 20. *l.* 6.) a Proportion duplicate to that which M F hath to I Q; that is, B F to Q R. Therefore the cylindrical Surfaces alſo have the ſame.

The ſame Figure: 5. Cylindrical Surfaces (B M, R I) have betwixt themſelves a Proportion compounded of the Proportions of the Sides (F M, I Q, and the Diameters of the Baſes (B F, Q R,) as is manifeſt from 23. *l.* 6. and the 7th of this.

Fig. 24, 25. *l.* 12. 6. If cylindrical Surfaces (A R, F D) be equal, as the Diameter (A B) is to the Diameter (F N) ſo reciprocally (by 14. *l.* 6.) the Altitude (F H) will be to the Altitude (B R,) and converſly.

7. Laſtly, from the ſame firſt *Corollary* 'tis had the Meaſure of a cylindrical Superficies; to wit, if the Circumference of the Baſe be multiplied by the Altitude. As if the Altitude be of 20 Feet, the Circumference of the Baſe of 6; multiply 20 by 6, there ariſe 120 Square Feet for the cylindrical Superficies.

PROP. XII. Theorem.

THE Superficies of a right Cylinder is to the Baſe (A B N) as the Side of the Cylinder (C B) is to (B O) the fourth part of the Diameter of the Baſe.

Let GH be a mean Proportional betwixt BC the Heighth, and B D the Diameter of the Baſe, and conſequently (by *Lemma* before *Prop.* 11 of this) a mean Proportional betwixt A N and the double of B C. The Circle G P H, of the Radius G H, is (by the 11th of this) equal to the Curve cylindrical Superficies C D. But the Circle G P H hath to the Baſe of the Cylinder A B N a Proportion duplicate (by 2. *l.* 12.) to the Proportion of G H to A N; that is, the ſame which the double of B C hath to the Radius BA (by the Hypotheſis, and *Def.* 10. *l.* 5.) that is, the ſame which B C hath to B O, the fourth Part of the Diameter.

meter: Therefore the cylindrical Superficies also is to the Base ABN, as BC is to BO, the fourth Part of the Diameter, *Q. E. D.*

Corollary.

THE Superficies of a Cylinder which hath its Sides equal to the Diameter of its Base, is four fold of the Base. But if the Side be a fourth Part of the Diameter of the Base, the Superficies of the Cylinder will be equal to the Base. Both these are manifest from the Proposition.

PROP. XIII. Theorem.

A Circle, whose Radius (OL) is a mean Proportional betwixt the Side (BC) of a right Cone, and the Radius of the Base (AC) is equal to the conical Superficies. Fig. 9, 8.

Let regular Polygons, E F, I N, be understood to be circumscrib'd about the Circles A C G, O P L, and a Pyramid circumscrib'd about the Cone to be erected upon the Polygon E F.

Because, by the Hypothesis, A C, or A G, is to O L, as O L is to B C, the Proportion of A G to B C, will (*Defin.* 10. *l.* 5.) be duplicate to the Proportion of A G to O L. But as AG is to B C, so is the Triangle under AG, and the Circuit E F, to the Triangle under B C and the same Circuit E F. Therefore the Proportion of the Triangle under A G, and the Circuit E F, to the Triangle under B C, and the same Circuit, is also duplicate to the Proportion of A G to O L. But the Triangle under A G and the Circuit E F, is equal to the Polygon L F (by the 4th of this:) And the Triangle under B C and the same Circuit E F (by the 9th of this) is equal to the Superficies of the circumscribed Pyramid. Therefore the Proportion of the Polygon E F, to the Superficies of the Pyramid, is also duplicate to the Proportion of A G to O L. But the Proportion of the Polygon E F to the Polygon I N, which is, by the Construction, like to it, is (*per* 1. *l.* 12.) also duplicate to the Proportion of A G to O L. Therefore the Polygon

Polygon E F hath the ſame Proportion to the Superficies of the Pyramid, and to the Polygon I N, which conſequently are equal. In the ſame manner I might ſhew, that the Superficies of Pyramids, which may be circumſcribed about a Cone infinitely more and more, are always equal to Polygons which may be circumſcribed infinitely about the Circle O P L Wherefore ſeeing both the Surfaces of Pyramids (by the 10th of this) do at laſt end in the Surface of the Cone, and Polygons (by the 3d of this) in the Circle O P L, the Superficies of the Cone and the Circle O P L, ſhall likewiſe be equal. *Q. E. D.*

From this excellent Theorem a Circle is found which is equal to a conical Surface.

Corollaries.

Fig. 9, 8. 1. THE Superficies of a right Cone, is equal to a Triangle comprehended under the Side of the Cone [B C] and the Circumference of the Baſe [C G.]

Let O L, the Radius, be a mean Proportional betwixt A C and B C. Then becauſe (by the 7th of this) the Circumference C G is to the Circumference P, as the Radius A G is to the Radius O L; that is, by the Hypotheſis, as O L is to B C; the Triangle under the firſt, to wit, the Circumference C G, and under the fourth B C (as appears from 16. *l.* 6.) will be equal to the Triangle under the ſecond, to wit, the Circumference P, and the third O L; that is, (by the 5th of this) to the Circle O P L; that is, to the conical Superficies (by the 13th of this) B C D. *Q. E. D.*

From this *Corollary* it appears that conical Surfaces have the ſame Properties with Triangles. And ſo it follows.

Fig. 20, 21. *l.* 12. 2. That the conical Superficies [B A F, Q X R] having their Sides [B A, Q X] equal, are betwixt themſelves as the Diameters of their Baſes [B F, Q R] And,

Fig 23, 24. *l.* 12. 3. Thoſe which have equal Baſes, C F T, A Z B, are betwixt themſelves as their Sides [C F, A Z.] And,

Fig. 20, 21. *l.* 12. 4. Thoſe conical Superficies [B A F, Q Z R] which are like, have betwixt themſelves a Proportion duplicate to that which is betwixt the Diameters of the Baſes. And,

The ſame Figure. 5. All conical Superficies whatſoever have betwixt themſelves a Proportion which is compounded of the Proportion

of the Sides [BA, QZ] and of the Diameters [BF, QR] which are in the Bafes. And,

6. Thofe which are equal have their Sides and the Diameters of the Bafes reciprocally proportional; and thofe which have them fo, are equal.

All which is demonftrated from *Corollary* 1. as above we deduced the *Corollaries* concerning the cylindrical Surface out of the firft *Corollary* there.

7. Laftly, we may meafure a right conical Surface, if we multiply the Side F C by half the Circumference of the Bafe. As if the Side be of 5 Feet, the Circumference of the Bafe of 20; multiply 5 by 10, and there will arife 50 Square Feet for the conical Superficies. The Demonftration is manifeft from the fame firft Corollary. *Fig.* 25. *l.* 12.

PROP. XIV. Theorem.

THE Superficies of a right Cone is to the Bafe, as the Side (BC) is to (AC) the Radius of the Bafe. *Fig.* 8, 9, *of this.*

Between the Side BC and AC, the Radius of the Bafe, let OL be a mean Proportional. Therefore the Proportion of BC to AC, is duplicate to the Proportion of OL to AC) *Defin.* 10. *l.* 5.) Now (by the 13th of this) a Circle of the Radius OL is equal to the conical Superficies CBD. But the Proportion of this to ACG, the Bafe of the Cone is (by 2. *l.* 12.) duplicate to the Proportion of OL to AC; and confequently is the fame with the Proportion of BC to AC. Therefore the Proportion of the conical Superficies CBD is to the Bafe ACG, as BC is to AC. *Q. E. D.*

Corollaries.

THE Superficies of a right Cone produced by an equilateral Triangle turned about the Perpendicular (KA) is double to the Bafe (QT.) *Fig.* 27.

For the Side KB is equal to BD, and confequently double to the half of it AB, which is the Radius of the Bafe.

2. *The*

Fig. 24. 2. *The Superficies of a Cone produced by a right-angled equicrural Triangle (EBD) is to the Base, as in a Square the Diameter is to the Side.*

For the Perpendicular B A being drawn, the right Angle B (by 26. *l.* 1.) is biſected, and conſequently A B D is half right. But A D B is alſo an half right Angle (by *Corol.* 11. *p.* 32. *l.* 1. Therefore D A, B A, are (by 6. *l.* 1.) epual; and conſequently B D is the Diameter of the Square A K, whereof A D is the Side. Now the ſame A D is the Semi-diameter of the Baſe P T, ſeeing the Perpendicular A B (by 26. *l.* 1.) biſects E D. From which, and this Fourteenth, the Corollary is manifeſt.

Fig. 24. 3. *The Superficies of the right Cylinder (G K) is to the Superficies of the right Cone (G B N,) as the Side of the Cylinder is to half the Side of the Cone.*

For the Superficies of the Cone, G B N, is to the Baſe M I, as the Side B N is to Q N, the Semi diameter of the Baſe (by the 14th of this) that is, as half the Side B N is to the fourth Part of the Diameter G N. But the Baſe M I (by the 12th of this) is to the Superficies of the Cylinder G K, as the fourth Part of the Diameter is to N K the Side of the Cylinder. By Equality of Proportion therefore the conical Superficies G B N is to the cylindrical Superficies G K, as half the Side of the Cone is to N K the Side of the Cylinder. *Q. E. D.*

A Lemma to what follows.

Fig. 10. IN a Triangle, as N P V, let there be drawn Q D parallel to N V.

I ſay, that the Rectangle under P N and N V is equal to the Rectangle under P Q, Q D, together with the Rectangle under N Q, and the two N V, Q D put together.

Draw N A perpendicular to the Side N P. and equal to N V; and the Rectangle N O being completed, let the Diameter P A be drawn. Then from Q let there be drawn

Q

QE parallel to N A, which may cut P A in B. Through B let C F be drawn parallel to N P. Becaufe A N is equal to NV, it is manifeft that Q B alfo is equal to Q D. (from *Corollary* 1. *p.* 4. *l.* 6.) Therefore the Rectangle O N is the Rectangle PNV, and F Q is P Q D. It remains that we prove that the Rectangles O B, E C, B N, are equal to the Rectangle under N Q, and the two N A, B Q; that is, to the Rectangle under N Q, and the two Lines N V, Q D. But that is manifeft; for the Rectangle under N Q and N A, Q B is equal (*per* 1. *l.* 2.) to thefe three Rectangles; that under N Q and C A (that is, the Space E C) and that under N Q and N C (that is, the Space B N) and that under N Q and Q B, that is again the Space B N, and confequently the Space O B, which (*per* 43. *l.* 1.) is equal to BN. The Propofition therefore is manifeft.

PROP. XV. Theorem.

IF a right Cone be cut by the Plane QS B, parallel to NZO; I fay, that the Circle GHM whofe Radius GH is a Mean betwixt Part of the Side NQ, and QD, NV taken together, the Radius's of the Circles QS B, NZO, is equal to the conical Surface intercepted betwixt the parallel Circles QS B, NZO. Fig. 11, 12.

Let G F be the Mean betwixt P N and N V. Likewife let G K be the Mean betwixt P Q and Q D; and let there be defcribed the Circles G F L, G K T. This (by the 13th of this) will be equal to the conic Superficies Q P B, and the other to the Superficies N O P. The Rectangle P N V (by the *Lemma*) is equal to the Rectangle P Q D, together with the Rectangle under N Q and N V Q D, taken together. But becaufe (by the Conftruction) G F is a mean Proportional betwixt P N, N V; the Rectangle P N V is equal to the Square of G F (by 17: *l.* 6.) And becaufe G K is (by the Conftruction) a Mean betwixt P Q, Q D, the Rectangle (by 17. *l.* 6.) P Q D is equal to the Square of G K: And becaufe G H, by the Hypothefis, is a Mean betwixt QN and Q D, N V taken together, the Rectangle (by 17. *l.* 6.) under Q N, and Q D, N V, taken together,

is equal to the Square of G H. Therefore the Square of G F is alſo equal to the Square of G H, and to that of G K. Therefore ſeeing Circles are betwixt themſelves (by 2. *l.* 12.) as the Square of the Radius's, the Circle G L F will alſo be equal to the two Circles G K T, G H M taken together. But (by the 13th of this) the Circle G L F is equal to the conical Superficies N P O. Therefore the conical Superficies N P O is alſo equal to the two Circles G K T and G H M. But Q P R, one Part of the Superficies N P O, is (by the ſame) equal to the Circle G K T. Therefore the remaining Part, which is comprehended betwixt the parallel Circles Z Z, S S, is equal to the Circle G H M. *Q. E. D.*

A Lemma to what follows.

Fig. 13: RIGHT Lines (B H, C G) which in the Circle intercept equal Arches (B C, H G) are parallel.

For let C H be drawn. Becauſe the Arches B C, H G are by the Hypotheſis, equal, the alternate Angles alſo (by 29. *l.* 3.) B H C, G C H, will be equal. Therefore (by 28. *l.* 1.) BH, and CG are parallel. *Q. E. D.*

PROP. XVI. Theorem.

Fig. 13. *LET there be inſcribed in a Circle a regular Figure of an even Number of Sides, and let it be equilateral; let E B be drawn from the Extremity of the Diameter unto B, the End of the Side next to the Diameter; and let the right Lines B H, C G, D F, join the Angles which are equally diſtant from A.*

I ſay, that the Rectangle contained under the Diameter A E, and the Subtenſe E B, is equal to the Rectangle of one Side of the inſcribed Figure A B, or B C, &c. and of all the joining Lines B H, C G, D F, taken together.

Draw C H, D G: Becauſe B H, C G, D F intercept (by 26. *l.* 3.) equal Arches B C, H G; C D, G F, theſe Lines

Lines (by the *Lemma*) will be parallel. By the ſame Arment B A, C H, D G, E F, are parallel. All the Triangles therefore (by 27, and 15. *l.* 1.) B A K, K H L, L C M, M G N, N D O, O F E, are Equi-angular. Therefore (by 4. *l.* 6.) as B K is to K A; ſo is H K to K L; and as H K is to K L; ſo is C M to M L; and as C M is to M L, ſo is G M to M N; and as G M is to M N, ſo is D O to O N; and as D O is to O N, ſo is F O to O E. Therefore (by 12. *l.* 5.) as one of the Antecedents B K, is to one of the Conſequents K A; ſo all the Antecedents B K, K H, C M, M G, D O, O F, (that is, all the joining Lines B H, C G, D F) are to all the Conſequents A K, K L, L M, M N, N O, O E, (that is, to the Diameter A E.) But (by 8. *l.* 6.) as B K is to A K, ſo is E B to B A. Therefore as all theſe together B H, C G, D F are to A E, ſo is E B to B A. Therefore (by 16. *l.* 6.) the Rectangle under B A on one Part, and all the joining Lines B H, C G, D F, on the other, is equal to the Rectangle which is under A E and E B. *Q. E. D.*

PROP. XVII. Theorem

LET there be inſcribed in D A F a Segment of a Circle, whoſe Baſe D F is perpendicular to the Diameter A O E, a Figure equilateral, and of an even Number of Sides; and let there be drawn, as in the foregoing, the Line E B. Fig. 14.

I ſay, that the Rectangle comprehended under E B and A O, part of the Diameter, which is the Axis of the Segment; is equal to the Rectangle, which is under one Side of the inſcribed Figure, and all the joining Lines B H, C G, &c. taken together with D O half the Baſe D F.

The Demonſtration is the ſame with that of the foregoing.

Lemma 1. *to what follows.*

LET there be inſcribed in the greateſt Circle of a Sphere a regular Figure, which hath its Sides meaſured by the number Four; and ſtands about the Axis A E; which Axis remaining unmov'd, let the Circle be turn'd round together with the Figure: Fig. 13.

I ſay,

I say, that there will be inscribed in the Sphere a Body contain'd under conical Superficies.

It is manifest (see *Defin.* 2. *l.* 12.) that B A, H A, likewise D E, E F, describe entire Superficies of right Cones. Then, because the Lines C B, G H, and G F, C D, being produced, do concur on both Sides in the same Point of the Diameter A E, which is in like manner to be drawn out, and cuts the joining Lines perpendicularly: It is also manifest that the said Lines C B, G H, &c. do describe Parts of right conical Surfaces which are intercepted betwixt the parallel Circles, which the Tops of the Angles B, C, D, design in the Spherical Superficies.

Lemma 2.

LET D A F be the greatest Section of a Segment of a Sphere whose Axis is A O. Let there be inscribed in this a Figure having all the Sides equal, the Base excepted, and let it be turn'd round about the Axis A O.

I say, that a Body contain'd under conical Superficies will be inscribed in the spherical Segment.

This is proved as the foregoing *Lemma*.

PROP. XVIII. Theorem.

Fig. 13. *LET the same Things be supposed which were in the first Lemma; and let the right Line (E B) be drawn from the Extremity of the Diameter unto the End of the Side next to the Diameter.*

I say, that a Circle, the Square of whose Radius (I) is equal to the Rectangle A E B, contained under the Diameter AE, and the Subtense E B, is equal to all the conical Surfaces inscribed in the Sphere.

That is a Circle whose Radius (I) is a mean Proportional betwixt A E and E B.

Because the right Lines B H, C G, D F, are equal to the right Lines B K, C M, D O, taken twice, (by 1. *l.* 2.) the Rectangle under one Side of the Figure inscribed in the greatest Circle, (to wit, under A B, or B C, or C D, or D E) and under all the joining Lines together, B H, C G, D F, is equal to the Rectangle under A B and B K, together with that which is under B C and the Compound of B K and C M,

C M, together with that which is under C D and the Compound of C M and D O, together with that which is under D E and D O; for ſo each of the Lines B K, C M and D O, are taken twice. But (by the 16th of this) the Rectangle under A B and all the joining Lines B H, C G, D F, taken together, is equal to the Rectangle A E B; that is, by the Hypotheſis, to the Square of I. Therefore the Square of I is equal to the Rectangles under A B and B K, and under B C and the Compound of B K and C M, under C D and the Compound of C M and D O, and under D E and D O. Now let P be a mean Proportional betwixt A B and B K; and Q a Mean betwixt B C and the Compound of B K and C M; and R a Mean betwixt C D and the Compound of C M, D O; S, a Mean betwixt D E and D O. The Squares therefore of P, Q, R, S, (by 17 *l.* 6.) are equal to the aboveſaid Rectangles. Wherefore ſeeing I have already ſhewed the Square of I to be equal to the ſame Rectangles, it muſt alſo be equal to the Squares of P, Q, R, S, together. Seeing therefore (by 2. *l.* 12.) Circles are betwixt themſelves as the Squares of their Radius's; the Circle deſcribed of the Radius I, will alſo be equal to all the Circles together whoſe Radius's are P, Q, R, S, as is manifeſt from 22. *l.* 6. But the Circles of the Radius's P and S, are (by the 13th of this) equal to the conical Superficies which the Sides A B, E D, have produced; foraſmuch as P is a mean Proportional betwixt A B the Side of the Cone, and B K the Radius of the Baſe; and S is a mean Proportional betwixt E D and D O; and the Circle of the Radius Q is (by the 15th of this) equal to that Segment of a conical Superficies which is intercepted betwixt the two parallel Circles of the Diameters C G, B H, becauſe Q is a Mean betwixt B C and the Compound of B K, C M: And for the ſame Cauſe the Circle of the Radius R is equal to a Segment of a conical Surface, which is intercepted betwixt the parallel Circles of the Diameters C G, D F. Therefore the Circle deſcribed from the Radius I, is equal to the conical Surfaces inſcribed in the Sphere taken all together. *Q. E. D.*

PROP. XIX. Theorem.

LET the ſame Things be ſuppoſed which were in the ſecond Lemma, and let the right Line E B be drawn from the Extremity of the Diameter (A E) to the End of A B, the Side next to the Diameter. *I ſay.*

I say, that a Circle, whose Radius is a mean Proportional betwixt (E B) and (A O) the Axis of the Segment, is equal to all the conical Superficies inscribed in the spherical Segment D A F.

The Demonstration is altogether the same with that of the foregoing; only for *Proposition* 16. let *Proposition* 17. be cited.

PROP. XX. Theorem.

Fig. 15. *CONICAL Superficies inscribed in a Sphere, do at length end in the Superficies of the Sphere.*

Let there be given a Superficies as small as you will, as X. It is manifest that within the spherical Superficies A C E G, there may be given some other Concentrical there, which falls short of this by a Quantity less than X. Let A C E G, D P L M, be the greatest Circle of both, as cut with a Plane through the Center. Let there be drawn the Diameter A D E, and in D let N Q touch it. If the Arch A E be bisected in C, and again the Remainder be bisected, and so on, there will be left at last the Arch A B (as is manifest of itself) less than the Arch A N. If to this Arch the right Line A B be subtended, it is manifest, that it will not reach to the Circumference P D M L, and that it will be a Side of an Equilateral Figure of an even Number of Sides inscribed in the Circle C A G E, no Side whereof reacheth unto the Circumference P D M L. Wherefore if all be turn'd round about the Diameter A E, it is manifest that there will be inscribed in the exterior spherical Surface, conical Surfaces, which include the spherical Surface, which is concentrical to the other, and consequently, by *Axiom* 3. of this, are greater. Because therefore the spherical Surface D P L M falls short of the spherical Surface A C E G, by a Quantity less than the given one X; much more will the conical Surfaces fall short of the said spherical Surface A C E G by a Quantity less than the given one X, and (by *Defin.* 6. *l.* 12.) consequently will end in the Superficies A C E G. *Q. E. D.*

PROP. XXI. Theorem.

Fig 17. *CONICAL Superficies inscribed in a spherical Segment D A F, end in the spherical Superficies of the Segment itself.*

This

This may be demonſtrated by the ſame Reaſoning as the foregoing was.

PROP. XXII. Theorem.

IT was demonſtrated, Prop. 18. *that a Circle, whoſe Radius is a mean Proportional betwixt the Diameter AE, and the right Line EB, which is drawn from the Extremity of the Diameter unto the End of the Side AB, next to the Diameter, is equal to all the conic Superficies inſcribed in the Sphere.* Fig. 16.

I ſay, that this Circle (ſee Def. 6. *l.* 12.) *ends at length in a Circle, whoſe Radius is AE, the Diameter of the Sphere.*

For if more and more Sides be infinitely inſcribed in the greateſt Circle (which then being turn'd round about A E, produce conical Superficies) ; it is manifeſt, that the Side AB becomes at length leſs than any given right Line, and conſequently that the Subtenſe E B approaches to the Diameter A E within a Diſtance leſs alſo than any given one; from whence it comes to paſs that the Difference of thoſe A E, B E, becomes likewiſe leſs than any given one. Therefore much more ſhall the mean Proportional betwixt A E, B E, which is always greater than B E, differ from A E at length by a Defect leſs than any given one. Therefore the Circle alſo whoſe Semi-diameter is a Mean betwixt A E and B E, will at length differ from a Circle whoſe Semi-diameter is A E, by a Defect leſs than any given one whatſoever, that is, will end (*Def. 6. l.* 12.) in it. *Q. E. D.*

This, which is clear enough of it ſelf, there is no need to demonſtrate more operoſely.

PROP. XXIII. Theorem.

IT was demonſtrated, Prop. 19. *that a Circle, whoſe Radius is a mean Proportional betwixt EB and the Axis of the Segment AO, is equal to all the conical Superficies inſcribed in the ſpherical Portion DAF.* Fig. 17.

I ſay, that this Circle ends in a Circle, whoſe Radius is the right Line AD, drawn from the Vertex of the Segment unto the Circumference of the Circle DQFN, which is the Baſe of the Segment.

For becauſe it now appears from the foregoing Demonſtration that E B doth at length end in A E; it will alſo be manifeſt that the mean Proportional betwixt E B and A O doth at length end in the mean Proportional betwixt A E and A O, that is, (by *Coroll.* 2. *p.* 8. *l.* 6.) in A D itſelf. It is therefore manifeſt that the Circle alſo whoſe Radius is a mean Proportional betwixt E B and A O, doth end in the Circle of the Radius A D. *Q. E. D.*

A Lemma to the following Propoſition.

IF the Diameter of one Circle be double ſo the Diameter of another, the one Circle will be four-fold to the other.

This is manifeſt from *Prop.* 2. *l.* 12. and *Defin.* 10. *l.* 5.

PROP. XXIV. Theorem.

Fig. 16. *THE Superficies of every Sphere is four-fold of the greateſt Circle of the ſame Sphere.*

This moſt noble Theorem of *Archimedes*, we ſhall, from what goes before, expeditiouſly demonſtrate in this manner.

Let an ordinate Figure, the Sides whereof are meaſured by the number Four, be underſtood to be inſcribed in the greateſt Circle of a Sphere about the Diameter A E. Let this Figure be turn'd round about A E, and ſo produce conical Surfaces inſcribed in the ſpherical Surface, and let E B be drawn. It hath been demonſtrated above (18. of this) that all conical Surfaces inſcribed in a Sphere are equal to the Circle, the Square of the Radius whereof is equal to the Rectangle A E B, that is, whoſe Radius is a mean Proportional betwixt A E and E B. And this will always happen, although the Inſcription be infinitely continued. Wherefore ſeeing the inſcrib'd conical Surfaces (by 20. of this) will at length end in the ſpherical Surface, and the Circle whoſe Radius is a Mean betwixt A E and E B, will at length end (by 22 of this) in the Circle whoſe Radius is A E; the ſpherical Surface it ſelf alſo (by 2. of this) will be equal

to

to the Circle of the Radius A E, that is, by the foregoing *Lemma*, to four Times the greateſt Circle A C E G. *Q. E. D.*

He that ſhall read *Archimedes*, will find that the Way here uſed in demonſtrating this moſt noble Theorem, is much ſhorter and clearer than that of *Archimedes.*

Corollary.

FROM this admirable Theorem, whereby *Archimedes* hath purchas'd to himſelf an immortal Name amongſt the Geometricians, a Circle equal to a ſpherical Surface is obtain'd; that, to wit, whoſe Semi-diameter is the Diameter of a Sphere, or whoſe Diameter is double to the Sphere's Diameter.

Scholium.

WE are now well provided for the meaſuring of a ſpherical Surface, the chief amongſt all Curve ones. And it is perform'd theſe two Ways.

1. Let the greateſt Circle of the Sphere be meaſured, (according to *Schol. Prop.* 6. of this) and let it be multiplied by 4. As, if the greateſt Circle of the Orb of the Earth be found to contain 49,075,500 ſquare Miles, or more exactly, 49,081,250 ſquare Miles, then, according to this, 196,325,000 ſquare Miles are contain'd in the whole ſpherical Surface.

2 The Diameter of a Sphere multiplied by the Circumference of the greateſt Circle gives you the ſpherical Superficies. According to which, if the Earth's Diameter conſiſt of 7,853 Miles, and conſequently the Circumference of the greateſt Circle conſiſts of 25,000, the whole ſpherical Surface will be in the ſame Miles 196,325,000; for 7,853 $\times$25,000=196,325,000.

The Demonſtration appears from *Corol.* 1. *Prop.* 5. of this; for a Rectangle under the Diameter of a Sphere, and the Circumference of the greateſt Circle, is according to that *Corollary* four-fold of the greateſt Circle.

PROP. XXV. Theorem.

THE Surface of any ſpherical Portion whatever (as D A F) is equal to a Circle, whoſe Radius is the right Line (A D) drawn from the Vertex of the Fig. 17.

the Portion to the Circumference of the Circle (DQFN) which is the Basis of the Segment.

Let a Figure, Equilateral and of an even Nnmber of Sides, the Base being set aside, be imagin'd to be inscrib'd in the Section of the greatest Portion about the Axis AO; this Figure being turn'd round about A O, will inscribe conical Surfaces in the Portion. Let the right Line E B be drawn also as above (in 18 and 19 of this.) All the conical Surfaces now inscribed are equal (by the 19th of this) to the Circle whose Radius is a mean Proportional betwixt E B and the Axis of the Segment A O And this will always happen if the Inscription be infinitely continued. Wherefore seeing both the conical Surfaces inscrib'd in the Segment end at length (by 21. of this) in the spherical Surface of the Segment, and the Circle whose Radius is a Mean betwixt E B and A O, ends (by 23) in the Circle of the Radius A D; the spherical Surface of the Portion also D A F (by 2.) will be equal to the Circle of the Radius A D. *Q. E. D.*

This is another of the more noble Inventions of *Archimedes*, which, as the former, we have demonstrated in a much shorter and clearer Way, than he did.

PROP. XXVI. Theorem.

Fig. 18. *THE Superficies of a right Cylinder circumscrib'd about the Sphere (as the Cylinder HPSV) is equal to the Surface of the Sphere.*

And if a Cylinder and Sphere be cut by Planes perpendicular to the Axis (BG;) each Segment of the cylindrical Surface will be equal to each Segment of the spherical Surface.

Part I. Because the Side H P of the Cylinder is (by the Hypothesis) equal to P S, the Diameter of the Base; the cylindrical Surface H S will be (by *Coroll. p.* 12. of this) four-fold of the Base; that is, of the greatest Circle of the Sphere inscrib'd in the Cylinder; of which, seeing (by the 24th of this) the spherical Surface it self is also four-fold, this will be equal to the cylindrical Surface. *Q. E. D.*

Part II. Let the right Lines B O, G O, be drawn. Because the Angle B O G (by 31. *l.* 3.) is right, as being the Angle

Angle in the Semi-circle, and O C falls perpendicular from it upon BG; BO (by *Coroll.* 2. *p.* 8. *l.* 6.) will be a mean Proportional betwixt G B and B C, that is, betwixt I T and H I. Therefore the Circle of the Radius B O (by 11. of this) will be equal to the cylindrical Surface H T. But the ſame Circle is alſo (by the foregoing) equal to the Segment of the ſpherical Surface OBK. Therefore the cylindrical Surface H T, and the ſpherical O B K, are equal.

Then becauſe it is ſhew'd in the ſame manner that the cylindrical Surface H X is equal to the ſpherical Q B R, the remaining cylindrical Surface I X will be equal to the remaining ſpherical Surface Q O K R, which is intercepted betwixt two parallel Circles.

And from theſe the Propoſition is manifeſt of all Segments.

[Corollary. "Hence the Superficies of a Cylinder circumſcrib'd about a Circle is double to the Baſes.]

PROP. XXVII. Theorem.

THE Segments of a ſpherical Surface divided by parallel Circles have that Proportion amongſt themſelves, which the Segments (B C, C D, D A, A E, E F, F G) of that Diameter (B G) which is perpendicular to the parallel Circles have amongſt themſelves. *Fig.* 18.

It follows from the foregoing. For by that the Segments of the ſpherical Surface O B K, Q O K R, M Q R N, *&c.* are equal to the cylindrical H T, I X, L N, *&c.* But theſe (by 13. *l.* 12. have the ſame Proportion betwixt themſelves, which the Segments of the Axis B C, C D, D A, *&c.* have. Therefore thoſe alſo have the ſame Proportion. *Q. E. D.*

Scholium.

FROM hence the Proportion of Zones and Climates betwixt themſelves becomes known. For they are to one another as the Segments of the Axis, which are known from the Table of Sines. *Fig.* 19.

From the ſame alſo we learn to meaſure the Segments of a ſpherical Surface. For becauſe both the whole Surface of the Sphere is known from *Schol. Prop.* 24. and the Proportion

tion of the Segments, the ſame as that of the Parts of the Axis, is alſo given; it is manifeſt that each of the Segments become known

Now both the foregoing, and all the reſt of the Theorems which follow, are altogether ſingular and admirable, and well worthy that thoſe who are ſtudious of Geometry ſhould give all Diligence to underſtand them.

A Lemma to the following.

Fig. 19. IF a Plane (QN) touch a Sphere in (O) a right Line (AO) from the Center to the Contact, is perpendicular to the Plane.

Let QN, the touching Plane and the Sphere, be cut through the Contact with two Planes, which in the Sphere may produce the Circles OG, OD, but in the Plane QN, the right Lines CO, IO, which ſhall touch the Circles in O. Therefore by 18. *l.* 3. AO is perpendicular to both IO and CO, and conſequently by 4. *l.* 11. perpendicular to the Plane QN. *Q. E. D.*

PROP. XXVIII. Theorem.

Fig. 20, 22, 21. *EVERY Sphere is equal to a Cone* (ZO) *whoſe Altitude* (KO) *is equal to the Radius of the Sphere*; *and the Baſe* (Z) *equal to the Superficies of the Sphere.*

Let ſome Polyedral Body be underſtood to be circumſcribed about the Sphere, and let the ſolid Angles thereof be cut off by new Planes touching the Sphere. Which being done, there will ariſe another Polyedral Body containing the Sphere, but leſs than the former, and conſiſting of more Angles, and having a Surface compounded of more Tangent Planes in Number, but leſs in Magnitude. If the ſolid Angle of this Polyedrum be again cut off by new Tangent Planes, and the Angles of the third Polyedrum thence ariſing likewiſe, and ſo on for ever; it will come to paſs at length, that both the Polyedrum will exceed the Sphere by a Solid leſs than any given one whatſoever; and the Surface thereof compounded of Tangent Planes (which, as I ſaid, are endleſly leſs in Magnitude, and more and more in Number than they were before) will exceed the ſpherical Surface alſo by a Plane

Plane leſs than any given one whatever. Both which Things although they might be demonſtrated, yet becauſe they are of themſelves manifeſt enough, I ſhall for Brevity-ſake take for granted. Theſe Things being thus ſtated, we proceed.

The Polyedrum now deſign'd is compounded of Pyramids, the common Top whereof is the Center of the Sphere, and the Baſes are Tangent Planes, which conſtitute the Surface of the Polyedrum. And becauſe the right Lines drawn from the Center A unto the Contacts of each of the Planes, are (by the foregoing *Lemma*) perpendicular to each of the Planes; therefore the Heighth of all the Pyramids, whereof the Polyedrum conſiſts, will be equal; to wit, A B the Radius of the Sphere. If therefore the Plane X be ſuppoſed equal to the Surface of the Polyedrum itſelf, and upon it there be erected a Pyramid at the Heighth M N, which is alſo equal to the Radius of the Sphere AB; it is manifeſt (by 6. *l.* 12.) that all the abovesaid Pyramids, that is, the whole Polyedrum are equal to the Pyramid X N. After the ſame manner all the reſt of the Polyedrums containing the Sphere, which from the perpetual Abſciſſion of the ſolid Angles will ariſe one after another infinitely, are always equal to the Pyramids (repreſented by X N) the Altitudes whereof M N are the Radius of the Sphere; but the Baſes (X) equal to the Surfaces of Polyedrums encompaſſing the Sphere. Wherefore ſeeing at length both the Polyedrums (as I ſaid above) do end in a Sphere, and the Pyramids (X N) as I will ſhew by and by, do end in the Cone Z O; the Sphere alſo (by 1. of this) will be equal to the Cone. *Q. E. D.*

But that the Pyramids X N end in a Cone I thus ſhew: The Surfaces of Polyedrums end in the Surface of the Sphere, as it was taken for granted above. But the Baſes X of the Pyramids X N, are always ſuppos'd equal to the Surfaces of the Polyedrums; and Z the Baſe of the Cone, Z O is, by the Hypotheſis, equal to the Surface of the Sphere; therefore the Baſes X alſo will end in the Baſe Z; and conſequently ſeeing the Pyramids X N to be to the Cone, which, by the Hypotheſis, is of equal Heighth (by *Coroll. Prop.* 11. *l.* 12.) as the Baſe X is to the Baſe Z, the Pyramids alſo will end in the Cone.

The Demonſtration of this Propoſition and the following, is altogether diverſe from that which *Archimedes* made uſe of, which indeed is very ſubtil and ingenious, but prolix and difficult;

difficult; to which there are premised two Positions that are manifest, and eleven Propositions, besides others, not a few, on which they depend. But the Theorem itself, as propounded by *Archimedes*, is thus; Every Sphere is fourfold of a Cone, which hath a Base equal to the greatest Circle of the Sphere, and its Altitude equal to the Radius.

Scholium.

FROM this noble Theorem is deduced the Mensuration of the most noble of solid Figures. For if the sixth Part of the Diameter, or the third Part of the Semi-diameter be multiplied by the Surface of the Sphere, already known by *Schol. Prop.* 24. there will arise the Solidity of the Sphere.

Suppose the Superficies of the Earth be found to contain 196,325,000 square Miles, and let the third Part of the Semi-diameter consist of 1309 such Miles. Multiply the two Numbers together, the Product 256,989,425,000 will be the Number of the cubic Miles of the Earth's Solidity.

For seeing a Sphere (by this *Prop.*) is equal to a Cone whose Altitude is the Radius of the Sphere, and its Base the Surface of the same Sphere, and the Solidity of the Cone (by *Schol. Prop.* 6. of this) is produced from the third Part of the Altitude (that is, of the Radius of the Sphere) multiplied by the Base (that is, the Surface of the Sphere) the Sphere's Solidity also is obtain'd from the third Part of the Radius multiplied into the Superficies:

PROP. XXIX. Theorem.

Fig. 23. *EVERY Sector of a Sphere is equal to a Cone, whose Altitude is the Radius of the Sphere, and the Base the spherical Superficies of the Sector.*

First, let the Sector A E C G be less than an Hemisphere. Let a right-lin'd polyedral Body be understood to be circumscrib'd about the Sector. Now, if all the remaining Ratiocination be carried on after the same manner as was done in the foregoing, the Thing sought will be concluded in the same manner. This Thing alone will require to be shew'd, upon which indeed the whole Reasoning depends; to wit, that the Superficies of the Polyedrum, which is compounded of

of Planes on every Side, touching the Surface of the Sphere E C G, is greater than the Surface E C G. Which is done thus. Let another equal and like Surface be conceiv'd to be set to the Surface ECG encompass'd with touching Planes in the very same manner as the other is. There will now (by *Axiom* 3. of this) the whole Surface compounded of Planes, be greater than the whole spherical Surface. Therefore half the Surface compounded of Planes will also be greater than half the spherical Surface E C G.

Then let the Sector A E B G, be greater than an Hemisphere. Both Sectors taken together, are (by the foregoing) equal to a Cone whose Heighth is the Radius of the Sphere, and its Bases the whole Superficies; that is, (by 11. *l.* 12.) to two Cones, which have the same Heighth, but have their Bases equal to the Segments of the spherical Superficies E C G, E B G. But one of the Sectors A E C G, that which is less than an Hemisphere, is by Part I. equal to a Cone, whose Altitude is the Radius, and its Base the Surface ECG. Therefore the other Sector E A B G, is equal to the other Cone, whose Heighth is the Radius, and its Base the remaining spherical Surface E B G. *Q. E. D.*

Corollary.

SEEING (by 25. of this) the Superficies E C G is equal to the Circle of the Radius C G, and the Superficies E B G equal to the Circle of the Radius B G; the Sectors A E C G, and A E B G, will be equal to Cones, whose Altitude is the Radius of the Sphere, and their Bases, Circles of the Radius's C G and B G.

Scholium.

FROM these Things is deduced the measuring both of Sectors and Segments of Spheres; of Sectors (as appears from *Schol. Prop.* 6 of this) if the third Part of the Radius be multiplied by the spherical Surface of the Sectors, which is already known from *Schol. Prop.* 27. or by the Circle of the Radius C G or B G; and of Segments, if the Cone E A G be measured, and be taken away from the Sector, if it be less than an Hemisphere; but added thereto, if it be greater. *Fig.* 23.

The

Fig. 18. The Segment (M Q R N) which lies betwixt two Circles, whether parallel or not parallel, is meaſured; if the Segments Q B R and M B N already known, be ſubſtracted one out of the other.

PROP. XXX. Theorem.

Fig. 24. *AN Hemiſphere (E O B D) is double to the Cone (E B D) which hath the ſame Baſe and Altitude with it ſelf.*

The Cone, whoſe Baſis is the Hemiſpherical Superficies E O B D, and its Altitude the Radius A B, is to the Cone E B D (by 11. *l.* 12.) as Baſe is to Baſe; that is, as the hemiſpherical Surface E O B D, is to the greateſt Circle P T. Therefore ſeeing the hemiſpherical Superficies E O B D is double to the greateſt Circle (by 24. of this) the Cone alſo which hath the Superficies E O B D for its Baſe, and the Radius A B for its Altitude, is double to the Cone E B D. But (by 28. of this) the Hemiſphere is equal to a Cone which hath the Radius for its Altitude, and the hemiſpherical Superficies for its Baſe. Therefore the Hemiſphere is alſo double to the Cone E B D. *Q. E. D.*

PROP. XXXI. Theorem.

Fig. 25. *LET a Sphere be divided into two Segments I L B G, I S K G, by the Plane I Q G T which doth not paſs through the Center A; and let the Diameter B O K be perpendicular to the cutting Plane.*

As the Altitude O B of the Segment I L B G, is to the Radius of the Sphere A B; ſo let O K, the Altitude of the other Segment, be made to the other Line K N.

In like manner, as O K, the Altitude of the Segment I S K G, is to the Radius A K or A B, ſo let the Altitude O B of the other Segment be made to the other Line B D. Which Things being ſuppos'd, I ſay,

1. *Th*

1. *The Cones ING and IDG whose Altitudes are ON, OD, and IQGT, their common Base, are equal to the spherical Segments.*

2. *There is the same Proportion of the Segments as there is of the right Lines DO, NO.*

3. *The Segment ISKG is to the greatest Cone IKG inscribed in it, as NO is to KO; and the Segment ILBG is to the greatest Cone IBG inscrib'd in it, as DO is to BO.*

Part I. Let the Sphere and Cones be cut by a Plane through the Diameter BK. There will be produced in the Sphere the greatest Circle BLKG, and in the Cones the Triangles BIG, IKG. And because BOK, the Diameter is (by the Hypothesis) perpendicular to the Circle QT, IOB (by *Def.* 3. *l.* 11.) will be a right Angle. The Angle in the Semi-circle is also a right one (by 31. *l.* 3.) Because therefore in the Triangle BIK, there is drawn from the right Angle, IO perpendicular to the Base BK; BI will be to IO, as (by 8. *l.* 6.) BK to KI. Therefore the duplicate Proportion of BI to IO is equal to the duplicate Proportion of BK to KI; that is, (because BK, KI, KO by *Coroll.* 2. *p.* 8. *l.* 6. are three Proportionals) equal to the Proportion of BK to KO.

Then because OB is (by the Hypothesis) to BD, as OK is to the Radius AB; by Inversion it will be always thus, DB is to BO, as AB to OK; and by Permutation thus, DB is to BA, as BO to OK; and by Compounding thus, DA is to BA, as BK is to OK. Because therefore I have already shew'd the Proportion of BK to OK, to be duplicate to the Proportion of BI to IO, and consequently (by 2. *l.* 12.) equal to the Proportion betwixt the Circles describ'd by the Radius's BI, IO; DA will also be to BA, as the Circle of the Radius BI, to the Circle of the Radius IO. Therefore the Cone under the Altitude DA, and for the Base, the Circle of the Radius IO; that is, the Circle QT, is equal to the Cone under the Altitude BA (by 15. *l.* 12.) which hath for its Base the Circle of the Radius BI; that is, (by *Coroll.* 29. of this) the spherical Sector AIBG. Wherefore if the same Cone IAG be added, as well to the Sector AIBG, as to the Cone under DA, and the Circle QT, the Wholes will be equal, to wit,

the spherical Segment I L B G, will be equal to two Cones, whereof one is that which is under the Base Q T and the Altitude D A, and the other I A G is under the same Base Q T, and the Altitude O A. But these two Cones (by 14. *l.* 12.) make up the Cone I D G. Therefore the Segment I L B G will be equal to the Cone I D G. *Q. E. D.*

By the same Reasoning, the Segment I S K G will be equal to the Cone I N G, with this only Change, that the Cone I A G, which before was added, be now taken away.

Part II. This is manifest out of the first. For the Cones I D G and I N G are betwixt themselves (by *p.* 14. *l.* 12.) as are D O and N O. Therefore the Segments also I L B G, I S K G, equal to those Cones, are betwixt themselves, as the right Lines D O, N O.

Part III. This likewise is manifest from the first. For the Cone IDG is to the Cone I B G, (by the same) as D O is to B O. Therefore the Segment also I L B G, which is equal to the Cone I D G, is to the Cone I B G, as D O is to B O.

Scholium.

FROM the first Part of this Proposition there arises another Way of Measuring spherical Segments, and that a very easy one; if, to wit, the Cones I D G, I N G, be measured; which will be done, if the third Parts of the right Lines D O, N O be drawn into the Circle Q T.

PROP. XXXII. Theorem.

Fig: 24. *A Right Cylinder (G K) is both in Solidity and the whole Superficies to the Sphere about which it is circumscrib'd, as 3 to 2.*

Let B Q be the common Axis of the Sphere and Cylinder, and EBD the greatest Cone inscrib'd in the Hemisphere E O B D. Because the Cylinder E K (half of G K) is (by 10. *l.* 12.) triple to the Cone E B D, while the Hemisphere is double to the same Cone, (by 30. of this) it is manifest that the Cylinder E K is to the Hemisphere as 3 to 2. Therefore also the whole Cylinder G K, is to the whole Sphere Q E B D, as 3 to 2. Which was the first Part.

Then because the Side of the Cylinder K N is equal to G N the Diameter of the Base, its Superficies without the Bases will be four-fold (by *Coroll. p.* 12. of this) of the Base

Base M I, and consequently taken together with the Bases, that is, the whole Superficies of the Cylinder, will be sixfold of the Base M I, which is equal to the greatest Circle of the Sphere. But the Superficies of the Sphere is fourfold of that greatest Circle. Therefore the whole Superficies of the Cylinder G K is to the Superficies of the Sphere, as 6 to 4, or as 3 to 2. Which was the other Part.

Therefore a Cylinder is both in Solidity and the whole Superficies to the Sphere, about which it is circumscrib'd, as 3 to 2. *Q. E. D.*

Scholium.

IT is an Argument what a great Value *Archimedes* puts upon this Theorem, that he would have a Sphere inscrib'd in a Cylinder set upon his Tomb. And perhaps amongst so many other famous Discoveries, this chiefly and above all others pleas'd him for this Reason, to wit, because there was one and the same rational Proportion both of Bodies, and of the Surfaces which contain them. We have demonstrated a like Identity of Affection betwixt Rings, and the Surfaces of Rings, in the Fourth Book of our Cylindricks and Annularies, *Prop.* 13, 14, 15. And another famous Example of the same hath also offer'd itself to me in the Sphere itself. For I have found, that like as a Sphere is to a right Cylinder which encompasseth it (which will necessarily be Equilateral) as 2 is to 3, and this both in respect of Solidity and Surface; so likewise the Sphere hath to an Equilateral Cone encompassing it, that Proportion which 4 hath to 9; and this both in regard of Solidity and Superficies. From which this also follows, That the sesquialteral Proportion found by *Archimedes* in the Sphere and Cylinder, is contained in three Solids, Sphere, Cylinder and Equilateral Cone. The Demonstration of both which Things, with some other Theorems of my own, in which the wonderful Nature of the Sphere will more appear, I shall subjoin in the thirteen following Propositions.

PROP. XXXIII. Theorem.

THE Superficies of a Sphere is double to the Superficies of a square Cylinder inscrib'd in the same Sphere. Fig. 26.

Let AKL be the Square inscrib'd in the greatest Circle of a Sphere, from which turn'd round, there is described a

ſquare Cylinder; and let A L be drawn as a Diameter common to the Cylinder and Sphere. Becauſe the Square A L is (by 47. *l.* 1.) equal to the equal Squares A K, K L, it will be double to one A K. Therefore alſo the Circle of the Diameter A L, is (by 2. *l.* 12.) double to the Circle, whoſe Diameter is A K; to wit, to the Circle C N. But the Superficies of the Sphere is (by 24. of this) four-fold to the Circle whoſe Diameter is A L; for that is the greateſt Circle of the Sphere, ſeeing A L is the Diameter of the Sphere. Therefore the Superficies of the Sphere is eight-fold of the Circle C N: But becauſe L K, K A (by the Hypotheſis) are equal, the cylindrical Superficies A C L is (by *Coroll. p.* 12. of this) quadruple of the Circle C N. Therefore ſince the Superficies of the Sphere is eight-fold of the ſame Circle, it will be double to the cylindrical Superficies. *Q. E. D.*

PROP. XXXIV. Theorem.

Fig. 26. *THE Superficies of a Sphere hath that Proportion to the whole Superficies of a ſquare Cylinder inſcrib'd in it, which* 4 *hath to* 3.

Let the ſame Things be ſuppos'd which were in the foregoing Demonſtration. Becauſe, by the Hypotheſis, L K, the Side of the Cylinder, and A K the Diameter of the Baſe thereof, are equal, the cylindrical Superficies C L will be quadruple (by *Coroll. p.* 12. of this) to the Baſe C N, and conſequently the whole Superficies of the Cylinder is to both Baſes C N and S L, as 6 is to 2. But the Superficies of the Sphere is to both Baſes together, C N, S L, as 8 is to 2, ſeeing in the foregoing it was ſhew'd that it is to one Baſe as 8 to 1. Therefore the Superficies of the Sphere is to the cylindrical Superficies C L, as 8 is to 6, or 4 to 3. *Q. E. D.*

Corollary.

THE whole Superficies of a right Cylinder deſcrib'd about a Sphere, is to the whole Superficies of an Equilateral Cylinder inſcrib'd, as 2 is to 1. For the circumſcrib'd is to the ſpheric Superficies as 12 is to 8 (by 32. of this.) But the Spheric is to the Inſcrib'd, as 8 is to 6 by this preſent Propoſition. Therefore the Circumſcrib'd is to the Inſcrib'd, as 12 is to 6, or 2 to 1.

PROP.

PROP. XXXV. Theorem

A Portion of any ſpherical Superficies whatever (as ILBG) hath the ſame Proportion to the Superficies of the greateſt inſcribed Cone, which (BG) the Side of the Cone hath to (GO) the Radius of the Baſe. *Fig. 26. or 25.*

Becauſe (by 25. of this) the Superficies of the Portion ILBG is equal to the Circle of the Radius BG; the Proportion thereof to QT, that is, to the Baſe of it ſelf and of the Cone, will be duplicate to the Proportion of the conical Superficies IBG to the ſame Baſe QT. Therefore it is manifeſt, (by *Definition* 10. *l.* 5.) that the Superficies ILBG is to the conical Superficies IBG, as the ſame conical Superficies IBG, is to the Baſe QT. Wherefore ſeeing the conical Superficies IBG, is to the Baſe QT, as BG (by 14th of this) is to GO, the Superficies of the Portion will alſo be to the conical Superficies IBG inſcrib'd in it, as BG is to GO. *Q. E. D.*

PROP. XXXVI. Theorem.

THE Superficies of the Hemiſphere (EOBD) hath that Proportion to (EBD) the Superficies of the greateſt right inſcribed Cone, which in a Square the Diameter hath to a Side; and that Proportion to the Superficies of a like Cone circumſcribed, as the Side in a Square hath to the Diameter. *Fig. 24.*

I. The Demonſtration of the firſt Part is manifeſt from the foregoing. For EOBD the Superficies of any Portion whatever, and conſequently of the Hemiſphere, is to the conical Superficies inſcrib'd, as BD is to DA. But BADK is a Square whoſe Diameter is BD, and the Side DA. *Fig. 6. l. 4.*

Part II. Let EBC be half of the Square circumſcrib'd about the Circle, (whoſe Center is O) which EBC being turn'd about the Axis AB, let there from thence be produced a Cone circumſcribed about the Hemiſphere. Now, becauſe the Square EC is (by 47. *l.* 1.) double to the Square EB or GI, the Circle of the Diameter EC alſo is (by 2. *l.* 12.)

l. 12.) double to the Circle whofe Diameter is G I, that is, to the Circle H G D I. But (by 24. of this) the Superficies of the Hemifphere included in the Cone E B C, is double to the fame Circle. Therefore the Circle of the Diameter E C is equal to the hemifpherical Surface Wherefore feeing the conical Superficies E B C is (by 14 of this) to the Circle of the Diameter E C, to wit, to its own Bafe, as the Side B E is to E O, the Radius of the Bafe; it will be alfo to the hemifpherical Superficies infcribed in it, as B E is to E O; that is, as the Diameter in a Square is to a Side. *Q. E. D.*

PROP. XXXVII. Theorem.

The fame Figure with Fig. 13. l. 5.

A Sphere hath the fame Proportion to a fquare conical Rhombus circumfcribed about it, both in refpect of the Solidity and Surface, which in a Square the Side hath to the Diameter.

Let the Square E B C F be circumfcrib'd about H G D I, the greateft Circle of a Sphere, from which Square, as turn'd round about the Axis B F, let a conical Rhombus encompaffing the Sphere be produced.

As E B, a Side of the Square (fee *Fig.* 6. *l.* 4.) is to the Diameter E C, even fo let S be made to R; (fee *Fig.* 13. *l.* 5.) and let this Proportion be continued through four Terms, S. R. Q. O; the Proportion then of S to O will be triplicate to the Proportion of S to R; that is, of E B to E C, and the Proportion of O to R will be duplicate to the Proportion of O to Q, or of R to S; that is, of E C to E B; and confequently (by 20. *l.* 6.) O is to R as the Square of E C is to E B; from whence (by *Schol. Prop.* 6. and 7. *l.* 4.) O is double to R. Thefe Things being thus fettled, let the Sphere E B C F be underftood to be circumfcribed about the conical Rhombus. Thus the Sphere H G D I will be to the Sphere E B C F (by 18. *l.* 12.) in the triplicate Proportion of the Diameter G I or E B to the Diameter E C; that is, (as I have already fhew'd) it will be as S to O But the Sphere E B C F is to the conical Rhombus infcribed in it (by 30 of this) as 2 is to 1; that is, (as I have fhew'd above) as O is to R. Therefore by Equality of Proportion, the Sphere H G D I is to the fame Rhombus which is defcribed about it, as S is to R; that is, as in a Square the Side E B is to the Diameter E C. Which was the firft Part. Then from the fecond Part of the foregoing, it appears, that the

Superficies

Superficies of the Hemisphere is to the Superficies of the Cone E B C, and consequently the Superficies of the whole Sphere is the Superficies of the whole Rhombus E B C F, as in a Square the Side is to the Diameter. Therefore the Sphere, as well in Solidity as in the Superficies, is to the square Rhombus E B C F, as in a Square the Side is to the Diameter. *Q. E. D.*

PROP. XXXVIII. Theorem.

THE Superficies of the Portion (B G K D) which contains an Equilateral Cone (B K D) is double to the Superficies of the same Cone. Fig. 27.

This is manifest from 35. For the Superficies of the Portion B G K D is to the inscrib'd conic Superficies (by 35. of this) as B K is to B A. But because the Cone B K D is suppos'd to be Equilateral, K B is equal to B D, and consequently double to B A. Therefore the Superficies B G K D is also double to the inscribed conical Superficies. *Q. E. D.*

PROP. XXXIX. Theorem.

THE Superficies of a Sphere is to the whole Superficies of an Equilateral Cone inscribed in it, as 16 *to* 9.

Let Z be the Center of the Sphere, and B K D the Equilateral Cone inscribed, and K Z A O the Axis common to the Sphere and Cone. If the Sphere and Cone be cut through this, there will be produced in the Sphere the greatest Circle, and in the Cone, the Equilateral Triangle B K D, one Side whereof will be the Diameter of the Basis of the Cone Q T. And because the Axis of the Cone K A is perpendicular to the Base Q T, B A K (*Def.* 3. *l.* 11.) will be a right Angle. Therefore the Square of BA is equal to the Rectangle K A O. (*Coroll.* 1. *p.* 17. *l.* 6.) Now because the Side of the Equilateral Triangle cuts off (*Coroll.* 5. *p.* 15. *l.* 4.) a fourth Part of the Axis A O, the Rectangle K A O, that is, the Square of B A, will be triple to the Square of A O (by 1. *l.* 6.) Wherefore seeing the Square of the Radius Z O is (*Coroll.* 3. *p.* 1. *l.* 2.) quadruple of the Square of A O, the Square of the Radius Z O will be to the Square of the Radius B A, as 4 is to 3. Fig. 27.

 Therefore

Therefore the Circle O B K D is alſo (by 2. *l.* 11.) to the Circle Q T, as 4 is to 3. Therefore four Circles, O B K D, that is, (by 24. of this) the whole ſpherical Superficies D G is to the Circle Q T, as 16 is to 3. But (*Coroll. p.* 14. of this) the Superficies of the Equilateral Cone B K D is to the Circle Q T, to wit, its own Baſe, as 2 is to 1; and conſequently the whole Superficies of the Cone B K D, that is, including its Baſe, is to the Baſe, to wit, the Circle Q T, as 3 is to 1, or 9 to 3. Therefore, ſeeing I have ſhew'd that the Superficies of a Sphere is to the ſame Circle, as 16 is to 3, the Superficies alſo of the Sphere D G will be to the whole Superficies of the Equilateral Cone, as 16 is to 9. *Q. E. D.*

Or otherwiſe thus:

BEcauſe, by *Corol.* 5. *Pr.* 15. *l.* 4.) the Side B D of the Equilateral Triangle cuts off a fourth Part of the Axis A O, the ſpherical Superficies B O D will be a fourth Part by 27. of this, and conſequently the Superficies B G D K, three fourth Parts of the Superficies of the whole Sphere. Wherefore if the whole Superficies be ſuppos'd to be 16, the Superficies BGKD will be 12. But (by the foregoing) the Superficies B G K D is double to the conical Superficies B K D, and conſequently is to it, as 12 to 6. Therefore the whole Superficies of the Sphere is to the conical B K D, as 16 is to 6. Then becauſe the Superficies of the Cone BK D (as being Equilateral) is (by *Corol.* 1. *Pr.* 14. of this) double to the Baſe Q T, it is manifeſt. that the conical Superficies B K D (to wit, without the Baſe) is to the whole Superficies of the Cone, as 2 is to 3; that is, as 6 to 9. Therefore by Equality of Proportion, the whole Superficies of the Sphere is to the whole Superficies of the Equilateral Cone inſcrib'd, as 16 to 9. *Q. E. D.*

PROP. XL. Theorem.

Fig. 28. *THE Superficies of the Sphere bears the Proportion to the whole Superficies of an Equilateral Cone circumſcribed about it, that* 4 *doth to* 9.

Let there be circumſcrib'd about the greateſt Circle of a Sphere BPM, the Equilateral Triangle DOF, by which as turn'd round about the Axis OA B, let there be produced an Equilateral Cone, circumſcribed about the Sphere. And

let

let there also be circumscribed about the Equilateral Triangle DOF, the Circle N D L O F, which, as is manifest, is concentrical to the former; and let the Axis OAB be produced to N. Because BN is a fourth Part of the Axis O N, as is manifest from *Corol.* 5, *Pr.* 15. *l.* 4.) O N is double to B K. Wherefore the Proportion betwixt Circles being duplicate (by 2. *l.* 12.) of the Proportion of the Diameters, the Circle BP M will be to the Circle N D L O F, as 1 to 4. But it hath already been shew'd in the foregoing Demonstration, that the Circle N D L O F is to the Circle Q T, the Base of the Equilateral Cone inscrib'd in the Sphere F L, as 4 is to 3. Therefore, by Equality of Proportion, the Circle B P M is to the Circle Q T, as 1 is to 3. But the whole Surface of the Cone D O F is (by *Corol.* 1. *Pr.* 14. of this) triple to Q T. Therefore the whole Superficies of the Cone is nine-fold of the Circle BP M. Wherefore seeing the Superficies of the Sphere T P is quadruple (by 24. of this) of the same Circle B P M. the whole Superficies of the Equilateral Cone D O F is to the Superficies of the Sphere to which it is circumscrib'd, as 9 is to 4. *Q. E. D.*

Corollary 1. " From this Demonstration it is manifest " that the Axis BO, of an Equilateral Cone, circumscrib'd " about a Sphere, is one and a half of the Diameter of the " Sphere EK, or as 3 to 2.

2. " That Q T, the Base of the Cone D O F, is also " one and an half of both Bases of the Cylinder circum- " scrib'd about the same Sphere. For Q T is to B P M, " as 3 to 1. Therefore Q T is to BPM twice, as 3 is to 2.

" 3. That the Superficies of the Cone D O F is one and " an half of the Superficies of the Equilateral Cylinder cir- " cumscrib'd about the same Sphere. For that * is double " to Q T, while this is quadruple to B P M †. Therefore " the conical Superficies will be to the cylindrical, as twice " 3 to four times 1; that is, as 6 to 4, or as 3 to 2.

* *Per Corol.* 1. *p.*14. *of this.*

† 24 *and* 26 *of this.*

PROP. XLI. Theorem.

Fig. 28:

THE whole Superficies of an Equilateral Cone circumscribed about a Sphere, is quadruple to the whole Superficies of a Cone inscribed in the same Sphere.

By the foregoing, the whole Superficies of the Equilateral Cone DOF circumscrib'd, is to the Superficies of the Sphere,

as 9 to 4; and the Superficies of the Sphere is to the whole Superficies of the inscribed Cone S K T, as 16 to 9 (by 39. of this.) Therefore by Permutation of Equality of Proportion, the whole Superficies of the circumscribed Equilateral Cone is to the whole Superficies of the Equilateral inscribed, as 16 to 4, or as 4 to 1. *Q. E. D.*

PROP. XLII. Theorem.

Fig. 29. *A Sphere hath that Proportion to B K C, an Equilateral Cone inscribed in it, which 32 hath to 9.*

Let the Sphere and Cone be cut by a Plane passing through the common Axis KO, producing in the Sphere the greatest Circle O F K I, and in the Cone the Equilateral Triangle B K C. Then a Plane being drawn thro' the Center A, perpendicular to OK, let the Hemisphere F G K I be cut off, in which let the greatest Cone F K I be understood to be inscribed. Now because (by *Cor.* 5. *p.* 15. *l.* 4.) the Side B C of the Equilateral Triangle cuts off a fourth Part of the Axis O K, P K will be to A K, as 3 to 2, that is, as 9 to 6. But the Base Q T is to the Circle O F K I, that is, to the Base N D, as 3 to 4, that is, as 6 to 8, as appears from what was demonstrated, *Prop.* 39. Wherefore seeing the Proportion of the Cone B K C to the Cone F K I, is (by *Schol.* 2. *p.* 15. *l.* 12.) compounded of the Proportion of the Altitude PK to the Altitude AK (that is, of the Proportion of 9 to 6) and of the Proportion of the Base Q T to the Base N D (that is, of the Proportion of 6 to 8) the Cone B K C will be to the Cone F K I, as 9 to 8. Wherefore seeing (by 30 of this) the Sphere C G is quadruple of the Cone F K I, the Equilateral Cone B K C will be to Sphere C G, as 9 to 32. *Q. E. D.*

PROP. XLIII. Theorem.

Fig. 28. *AN Equilateral Cone circumscribed about a Sphere, is eight-fold of an Equilateral Cone inscribed in the same Sphere.*

Let S K T and D O F be the Equilateral Cones inscrib'd and circumscrib'd, and let O K B be the common Axis. Then let as well both the Cones as the Sphere be cut by a Plane

Plane passing through the Axis; their Sections will be two Equilateral Triangles, and the greatest Circle B P M. About the Triangle D O F likewise let there be understood to be describ'd the Circle N D O F, and let the Axis O K B be produced unto N. Now because the Side DF of the Equilateral Triangle doth (by *Cor.* 5. *p.* 15. *l.* 4.) cut off N B, the fourth Part of the Axis O N, it is manifest that O N is double to BK. In like manner, because the Side S T of the other Equilateral Triangle cuts off B C, the fourth Part of the Axis B K, N O will be to B O, as F K is to C K; and by changing, as N O is to B K, so is B O to C K. But N O is double to B K. Therefore B O is likewise double to C K. Therefore because of the Similitude of the Triangles, DOF, SKT, D F and S T also, to wit, the Diameters of the conical Bases will (by 4. *l.* 6.) be in a double Proportion betwixt themselves. Wherefore seeing the Cones D O F, S K T, be like, and consequently (by 12. *l.* 12.) their Proportion is triplicate to the Proportion of the Diameters D F and S T; which is that of 2 to 1, the Cone D O F will be to the Cone S K T, as 8 to 1. *Q. E. D.*

PROP. XLIV. Theorem

A Sphere hath the same Proportion both in respect of Solidity and Surface to the Equilateral Cone D O F circumscribed about it, which 4 hath to 9. Fig. 28.

The Sphere T P is (by 42. of this) to the Equilateral Cone S K T inscribed in it, as 3 is to 9. But (by the foregoing) S K T, the Equilateral Cone inscribed, is to D O F, the Equilateral Cone circumscribed, as 1 is to 8, that is, 9 to 72. Therefore by Equality of Proportion, the Sphere T P is to D O F, the Equilateral Cone circumscribed, as 32 is to 72, that is, as 4 to 9. But in *Prop.* 40. we demonstrated that the Superficies of a Sphere is to the whole Superficies of an Equilateral Cone circumscribed, as 4 is to 9. Therefore a Sphere, both in Solidity and Superficies, is to an Equilateral Cone circumscribed about it, as 4 is to 9. *Q. E. D.*

That therefore which *Archimedes* was surpris'd at in a Sphere and Cylinder encompassing it, we have also now demonstrated in a Sphere and an Equilateral Cone encompassing it, to wit that there is the same rational Proportion of the Solidities betwixt themselves, which there is of the Surfaces.

For

For as he found that the Sphere is to the Cylinder, as well in Solidity as Superficies, as 2 to 3; so we have now taught, that the Sphere is, in respect both of Solidity and Surface, to an Equilateral Cone encompassing, as 4 to 9.

But from hence we shall, without much Labour, demonstrate, that the very Proportion, to wit, the Sesquialteral, which *Archimedes* shew'd to be betwixt the Sphere and Cylinder; is continued by the Equilateral Cone circumscribed both in the Solidity and Superficies; and so we shall put an End to the present Work.

PROP. XLV. Theorem.

See the Figure prefixed to this Treatise.

AN Equilateral Cone circumscribed about a Sphere, and a right Cylinder in like manner circumscribed about the same Sphere, and the same Sphere it self, continue the same Proportion; to wit, the Sesquialteral, as well in respect of the Solidity as of the whole Superficies.

For by 32. of this Book, the right Cylinder G K encompassing the Sphere, is to the Sphere, as well in respect of Solidity, as of the whole Superficies, as 3 is to 2, or as 6 to 4. But by the foregoing, the Equilateral Cone B A D circumscribed about the Sphere, is to the Sphere in both the said respects, as 9 is to 4. Therefore the same Cone is to the Cylinder, both in respect of Solidity and Surface, as 9 is to 6. Wherefore these three Bodies, a Cone, Cylinder and Sphere, are betwixt themselves, as the Numbers 9, 6, 4, and consequently continue the sesquialteral Proportion. *Q. E. D.*

[*PROP. XLVI.*]

THE same sesquialteral Proportion holds betwixt an Equilateral Cone and Cylinder circumscribed about the same Sphere, in respect of their whole Surfaces, their simple Surfaces, their Solidities, Altitudes and Bases.

This Proposition is manifest as to the whole Surfaces and Solidities from the foregoing; as to the simple Surfaces, from Cor. 3. Pr. 40. *of this; as to their Altitudes and Bases, from* Cor. 1. *and* 2. *of the same 40th Proposition.*]

FINIS.

AN

APPENDIX

OF

PRACTICAL GEOMETRY.

In TWO PARTS.

The FIRST contains an *Easy*, *Brief* and *Independent* Demonstration of certain Select and most Useful *Propositions* in *EUCLID*.

The SECOND contains a *Synopsis* of all the *Problems* in the preceding Book; with a a *Supplement* of sundry choice *Problems* in Practical Geometry.

For the Use of young Students in the Mathematicks.

By S. F.

DUBLIN:

Printed in the Year MDCCLIII.

APPENDIX.

PART I.

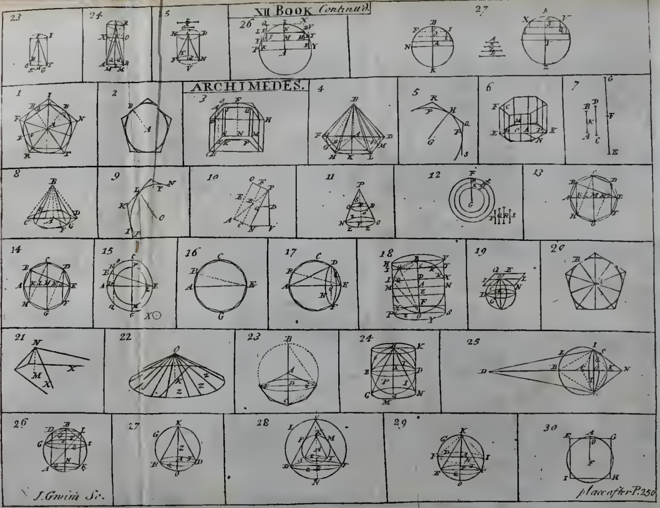
WE often find that EUCLID, for the demonstrating one *important* Propoſition, hath made uſe of a long Chain of others, which have no other End, but to demonſtrate that *principal* One: If we can *all at once* demonſtrate thoſe *capital* Propoſitions without ſuch a *long Series* of preparatory Demonſtrations; we ſhall doubtleſs retrench many uſeleſs Things, gain Time, and render this Appendix of ſome Service to the young *Student*.

There are two primary Propoſitions, of which *Des Cartes* writes in a Letter not yet printed, as hinted in *Page* 65. of Dr. PELL's *Algebra*; *In ſearching the Solution of Geometrical Queſtions, I always make uſe of Lines, Parallel and Perpendicular, as much as poſſible,* (i. e. *as many Lines as are uſeful*) *and I conſider no other* Theorems, *but the* 47*th*, L. 1 *and* 4*th*, L. 6.
of

of Euclid; *and I am not afraid to ſuppoſe many unknown Quantities, that I may reduce the propoſed Queſtion to ſuch Terms, as to depend on no other* Theorems, *but theſe two.*

Obſerve in how great Eſteem the ingenious *Des Cartes* held theſe two *Theorems*; to the firſt of which, moſt of the preceding Propoſitions in EUCLID are preparative, and which we ſhall Eſſay to demonſtrate *independently* of any of them.

EUCLID.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EUCLID.

BOOK I.

PROP. XLVII. Theorem.

IN every right-angled Triangle (A B C) the Square of the Side (A C) which is opposite to the right Angle, is equal to the two Squares of the other Sides A B and B C) taken together.

In this are two Varieties; First, when the two Sides AB and B C are equal.

On the Side A C erect the Square A C D E, on the Side A B erect the Square A B D G, on the Side B C erect the Square B C E F *Fig. 1.*

Demonstration. 'Tis evident at first View of *Figure* 1. that one half of the Square of A B or B C, is an exact quarter of the Square of A C, consequently four halves of the Squares of A B and B C, which are both their whole Squares, are equal to four quarters of the Square of A C, which is the whole Square of A C. *Q. E. D.*

Secondly, when the Sides A B and B C. are unequal. *Fig. 2.* Make two equal Squares B D E F. B D G H on the Line B D, equal to A B and B C, the two less Sides of the given Triangle; in the first, on A C, erect the Square A C I K; in the other, on A B and B C, erect the Squares A D B L =□ B C and B C H M =□ A B; and last of all, draw the Lines A C and B G.

Demonstration. First, 'Tis evident from the Construction, that the Squares B D E F and B D G H are equal.

Secondly, 'Tis obvious at first View, that the Square B D E F exceeds the Square of A C by four right-angled Triangles, whose Base is = A B and Perpendicular = B C, and consequently, by *Axiom* 7. equal to one another, and the given Triangle A B C.

Thirdly, 'Tis equally evident, B D G H exceeds the Squares of A B and B C, by four right-angled Triangles, whose Base is = A B and Perpendicular = B C, and consequently, by *Axiom* 7. equal to one another, and the given Triangle A B C, and also to the four right-angled Triangles aforesaid.

Fourthly, Wherefore, by *Axiom* 3. if from Equals, (the Squares B D E F and B D G H) be taken away Equals (the four right-angled Triangles in each Square) the Remains will be equal (the Square of AC in one Equal to the Squares of A B and B C in the other.) *Q. E. D.*

Or Algebraically thus:

Let the biggest Square made of the Sum of the Sides = S S

$$\left.\begin{array}{l}\Box AC = HH\\ \Box AB = BB\\ \Box BC = PP\end{array}\right\}\text{ I say, } HH = BB + PP.$$

1. $SS = HH + 2BC$, which $2BC = 4$ Triangles.
2. $SS = BB + PP + 2BC$, by $B + C \times B + C$.
3. $HH + 2BC = BB + PP + 2BC$.
4. $HH = BB + PP$. *Q. E. D.*

A like useful Theorem of signal Service in the whole Theory of Compound Motions, I shall subjoin.

In every Parallelogram the Sum of the Squares of the two Diagonals is equal to the Sum of the Squares of the four Sides.

To prove this by Trigonometry requires 21 Operations, by Analysis, or Algebra 15, which M. *de Lagny* has reduced to 7 Steps.

Howbeit

Howbeit the Reasonableness of this Theorem may appear Fig. 3.
from this single Consideration, that all Triangles on the same or equal Base, and betwixt the same Parallels, are equal, compared with the 12th and 13th of the second Book.

In this are two Varieties; First, When the Parallelogram is right angled.

Then the Proposition is evident from the 47th, I. *Euclid*, just now demonstrated.

Secondly, But when the Angles of the Parallelograms are oblique, draw the prick'd Lines AG, BF, CH, DE, which being Perpendiculars betwixt the same Parallels, are all equal: Also GB and DH, which are also equal to AF and EC, being perpendicular betwixt the equal Parallels, let each of the first $4 = y$, and each of the last $4 = z$; also $AD = BC$ call x; consequently $BE = x - z$.

Demonstration.

$$\text{4 Sides.}\left\{\begin{array}{l} \square\, BC = xx. \\ \square\, AD = xx. \\ \square\, AB = yy + zz. \\ \square\, DC = yy + zz. \end{array}\right. \qquad \text{2 Diag.}\left\{\begin{array}{l} \square\, BD = yy + xx + zz - 2xz. \\ \square\, AC = yy + xx + zz + 2xz. \end{array}\right.$$

$$\text{Sum } 2yy + 2xx + 2zz. = \text{Sum Dia. q. } 2yy + 2xx + 2zz.$$

Corollary. "Hence 'tis plain, the Square of the longer
" Diagonal exceeds the Sum of the Squares of the two
" contiguous Sides, exactly by as much as the Square of
" the shorter Diagonal wants thereof; that is to say, by
" the double Rectangle xz, whose Length is the longest
" Side, and Breadth equal to the Distance that the Perpen-
" dicular from the opposite Angle falls from one End of it,
" either within or without the Parallellogram.

EUCLID. Book VI.

PROP. IV. Theorem.

Triangles which are Equiangular to one another, are like or ſimilar, that is, have their Sides alſo that are oppoſite to the equal Angles proportional.

Fig. 4. This I take to be the ſame with the firſt Definition of the ſame Book.

And that ſimilar Triangles (A B C, A D E) have their correſponding Sides proportional, I ſhall illuſtrate from *Figure* 4.

Let the Triangle A B C be laid on the Triangle A D E, ſo will the Angle A, becauſe equal in both, exactly coincide, and the Line A C fall on the Line A E, and A B on A D, by *Axiom* 7. and becauſe the Angle C = Angle E, and the Angle B = Angle D, the Side B C will be parallel to D E, by *Defin.* 1. *Book* 6.

Then ſuppoſe A B a third of A D, make B F another third, and parallel to D E draw F G, then parallel to A D, draw GH and CI; by which Parallels, the Sides A E and DE will alſo be divided into three equal Parts, by *Axiom* 12. Therefore it will hold as,

AB : AD :: AC : AE		AC : AE :: AB : AD
4 12 5 15		5 15 4 12
that is,		that is,
AB : 3AB :: AC : 3AC		AC : 3AC :: AB : 3AB
AB : AD :: BC = DI : DE	and	CB = DI : DE :: AC : AE
4 12 3 9		3 9 5 15
that is,		that is,
AB : 3AB :: DI : 3DI		DI : 3DI :: AC : 3AC.

Q. E. D.

To

To Trisect an Arch of a Circle B C.

This has been by sundry Antient and Modern Geometricians accounted impossible to be done by the *Euclidean* Geometry, which makes use of only a Circle and strait Lines; howbeit, we will attempt it, and afterwards a Demonstration thereof

Quarter the Circle, and extend the Diameter B G to P; parallel to it, draw CX and CR parallel to EX; then take the Diameter B G in the Compasses, and move the Ruler on the Point C, till F P be exactly equal to B G, then draw C F O P, so will G O be a third of B C. *Fig. 5.*

Demonstration.

Through the Center draw O E Z. Now because E O is Radius, and FP the Diameter, and the Angle FEP is right; therefore the Lines F O, E O, P O, are all equal, and also the external Angle F O E = 2 P E O, the two internal opposite Angles, or 2 ZEB, which is Vertical, and consequently equal to P E O. But F O E, that is, C O Z, being in the Periphery, is measured by half the Arch C Z; wherefore B Z, which is the Measure of half the Angle C O Z, is a fourth of the whole Arch CZ, and consequently a third of B C; and therefore G O = B Z is also a third of B C. *Q. E. D.*

Note, If the Arch to be trisected be greater than a Quadrant, then trisect its Complement to 180, and the third of this Complement, taken from 60 Degrees, always leaves the third of the Arch required.

Least the foregoing Demonstration should appear too concise to some, I will attempt it after another Manner, from the following *Lemma*.

" That the Measure of the Angle C P B, at a Point
" without the Circle is equal to half the Difference betwixt
" the intercepted Arches C B and O G, or to the Differ-
" ence betwixt half their Sum C O Z, and the less Arch
" OG, which is the same.

For B Z=O G, becaufe Vertical Angles, by 15. *l.* 1. then is C Z the Sum of the intercepted Arches C B and O G, and C O Z, the Angle at the Circumference, the Meafure of half the faid Sum, by 20. *l.* 3. which C O Z, being external, is equal to the two internal and oppofite Angles O E P and O P E, by 32. *l.* 1. but the Meafure of O E P is O G, the lefs Arch: Now, if C O Z=O G, and the Angle O P E=CPB. I fay, C P B muft be equal to the Difference betwixt C O Z and O G. *Q. E. D.*

Corollary. " In the faid Triangle E O P, if the Angles " at E and P be both equal. then will O G be a third of " B C; becaufe O G will be half of C O Z, or a fourth of " the whole Arch C Z. and confequently a third of B C " $=\frac{3}{4}$ C Z, for from the whole C Z, take away B Z$\frac{1}{4}$, " remains B C=$\frac{3}{4}$. It remains only to prove the Angles " E and P are equal.

Becaufe F E P is a right Angle by Conftruction, the Center of the Semi-circle F E P will fall in O, which bifects F P, the Diameter, which is double E O, the Radius; confequently all three, F O, E O, P O, are equal Radius's of the Semi-circle F E P aforefaid; and becaufe E O and P O are equal, the Angles at the Bafe E and P, by 5. *l.* 1. are equal.

A Synopfis *of the moft ufeful and famous* Propofitions *in* Euclid's *firft Six Books*,

BOOK I. has *Propofitions* 32, 35, 37, 41, 44, 45, 47.
II. ——— 4, 5, 6, 12, 13, 14.
III. ——— 16, 20, 21, 22, 31, 32, 35, 36.
IV. ——— 10, 11, 12, 13, 14, 15, 16.
V. ——— { 15, 16, 17, 18. or the foregoing Compend. thereof.
VI. All the firft 6, 8, 13, 14, 16, 19, 31.

Ex-

Extraordinary Propositions *not to be met with in* EUCLID, *whose Demonstrations are omitted on purpose to Exercise the Genius of young Mathematicians.*

1. IN every Triangle the Rectangle of any two Sides is equal to the Rectangle of the Perpendicular from the said Angle, and the Diameter of the circumscribing Circle, *i. e.* as Perpendicular : one Side : : other Side : Diameter.

2. The Area of any Triangle about a Circle, is equal to the Rectangle of the Semi-diameter, and half the Sum of the Sides.

3. An Hexagon inscribed is a Mean betwixt a Trigon inscribed, and a Trigon circumscribed, *Et sic de paribus.*

4. In any Triangle the Difference of the Squares of two Sides, is equal to a Rectangle of the Base, and that Segment of the Base, which parted, in the middle of the other Part the Perpendicular falls.

5. The Square of the Mean, and the Square of half the Difference of the Extremes together, is equal to the Square of the half Sum of the Extremes.

6. If in a Circle two Lines be inscribed, intersecting each other, the Rectangles of the Segments of each Line are equal; and the Angle at the Point of Intersection is measured by half the Sum of its intercepted Arches.

7. If to a Circle two right Lines be adscribed from a Point without the Rectangles of each Line, from the said Point to the Convex and Concave are equal, and the Angle at the Point is measured by half the Difference of the intercepted Arches.

8. If in a Circle three right Lines shall be inscribed, one of them cutting the other two, then the Rectangles of the Segments of each Line so cut, are directly proportional to the Rectangles of the respective Segments of the said cutting Line.

9 If a plain Triangle be inscribed in a Circle, the Angles are one half of what their opposite Sides do subtend; and if it hath one right Angle, the longest Side of that Triangle shall be the Diameter of the Circle.

Easy Equations *arising by comparing two most of the* Propositions *of the Second* $n=3$ $z=10$ *and* $x=4$.

The Reason.	Numb Equat.	In Species.	In Numbers.
	1	$z=m+n$	$10=7+3$
	2	$x=m-n$	$4=7-3$
$1-n$	3	$m=z-n$	$7=10-3$
$1-m$	4	$n=z-m$	$3=10-7$
$2+n$	5	$m=x+n$	$7=4+3$
$2-x$	6	$n=m-x$	$3=7-4$
$1+2$	7	$2m=z+x$	$14=10+4$
$1-2$	8	$2n=z-x$	$6=10-4$
$7 \div 2$	9	$m=\frac{z+x}{2}$	$7=\frac{10+4}{2}$
$8 \div 2$	10	$n=\frac{z-x}{2}$	$3=\frac{10-4}{2}$
$7-x$	11	$z=2m-x$	$10=14-4$
$7-z$	12	$x-2m-z$	$4=14-10$
$8+x$	13	$z=2n+x$	$10=6+4$
$13-2n$	14	$x=z-2n$	$4=10-6$
$2\times x$	15	$xx=xm-xn$	$16=28-12$
3×4	16	$mn=zz+mn-zm-zn$	$21=100+21-70-30$
5×5	17	$mm=xx+2xn+nn$	$49=16+24+9$

unequal Quantities and Numbers, whereby Book of Euclid *are invented. Let* $m = 7$

The Reaſon	Numb. Equat.	In Species.	Propoſitions of the Second Book.
$1 \times b$	18	$zb = bm + bn$	1
$1 \times z$	19	$zz = zm + zn$	2
$1 \times m$	20	$zm = mm + mn$	3
$1 \times n$	21	$zn = nn + mn$	
1×1	22	$zz = mm + 2mn + nn$	4
2×2	23	$xx = mm - 2mn + nn$	
7×8	24	$4mn = zz - xx$	
$24 \div 4$	25	$mn = \frac{zz}{4} - \frac{xx}{4}$	
$25 + \frac{xx}{4}$	26	$\frac{zz}{4} = mn + \frac{xx}{4}$	5
1×2	27	$zx = mm - nn$	
$27 + nn$	28	$zx + nn = mm$	6
3×3	29	$mm = zz - 2zn + nn$	
$29 + 2zn$	30	$mm + 2zn = zz + nn$	7
$24 + xx$	31	$4mn + xx = zz$	8
$22 + 23$	32	$zz + xx = 2mm + 2nn$	10
$32 \div 2$	33	$\frac{zz + xx}{2} = mm + nn$	9
		And ſo infinitely.	

PRACTICAL GEOMETRY.

PART II.

AS to the Demonstration of these *Practical Problems*, I've purposely omitted them, that the young Student may exercise his own Genius, in making Application of, and recollecting what he has already learned from the foregoing Books of *EUCLID*; to awaken the Mind, to whet the Appetite of our Mathematical Student, and to amuse and improve him in easy and practical Problems of Geometry, is the main Design of this Appendix.

And lest we go over what is already done, I think it not amiss to give a *Synopsis* of all the *Practical Problems* both in the *Propositions* and *Corollaries* of the foregoing Book, referring to the Book and Proposition where they may be found.

In the First Book.

PROPOSITION 1. On a given Line to make an Equilateral Triangle.

To measure inaccessible Distances, see 4. and 26. also *L. 6. Prop.* 8.

2. To draw a Line equal to a given one.
3. From a greater to cut off a less.
4. To play at Billiards.
6. To measure accessible Altitudes, see 33. also *L.* 6. 4.
9. To bisect an Angle.
10. To bisect a Line.
11. To erect a Perpendicular, and *L.* 3. 31.
12. To let fall a Perpendicular.
15. That Rays of Light reflected, take the shortest Course.
19. A Globe can rest no where, but in the Point it touches the Earth.
22. To make a Triangle of three Lines given.

22. To make an Angle equal to a given one.
To meafure a given Angle.
To lay down an Angle of any Number of Degrees.
27. To meafure the Compafs of the Earth, fee *L.* 2. 6.
31. To draw Parallels.
32. To determine the Parallax of the Stars.
To find the Number of right Angles contain'd in the Angles of any right-lin'd Figure.
33. The Demonftration of Compound Motions.
34. To divide the Area of a Parallelogram.
36. Figures of equal Compafs may have different Area's.
38. To divide the Area of a Triangle; that Bodies move equal Area's in equal Times,
40. And are urged by a Centripetal Force.
41. To find the Area of a Triangle, *L.* 2. 13.
42. To make a Parallelogram with an Angle equal to a given one, and equal to a Triangle given.
44. On a given Line, to make a Parallelogram equal to a Triangle given, and to have an Angle equal to one given.
To demonftrate Geometrical Divifion.
45. On a given Line, and with a given Angle, to make a Parallelogram equal to any ftrait-lin'd Figure.
To find how much one ftrait-lin'd Figure exceeds another.
46. To make a Square on a given Line.
47. To add any Number of Squares.
To take a lefs Square out of a greater; any two Sides of a right-angled Triangle, to find the third.
The Origine of the Table of Sines, Tangents and Secants. See *L.* 3. 3. *L.* 4, &c.

In the Second Book.

11. To cut a Line in extreme and mean Proportion, *L.* 6. 30.
14. To find a Square equal to any right-lin'd Figure.

In the Third Book.

1. To find the Center of a Circle.
12. To find the Point where two Circles touch each other.
16. To demonftrate the infinite Divifibility of a ftrait Line. See *L.* 1. 47.
17. To draw a Tangent to a Circle.

20. To

20. To demonſtrate the Sides of Triangles, are in ſuch Proportion as the Sines of their oppoſite Angles.
    To meaſure the Diſtance of the Sun or Moon.
21. How the ſame Line, at different Diſtances, may appear of the ſame Length.
22 About what Quadrangle a Circle can or cannot be deſcribed, *L*. 6. 4.
25. To perfect an Arch into a Circle.
30. To biſect a given Arch.
31. To prove whether a Square be true.
33. On a Line to form a Segment capable of any given Angle.
34. From a Circle, to cut a Segment containing any given Angle.
35. Certain Geographical Paradoxes ſolv'd; to draw a Circle through any two Points in another given Circle, which ſhall diametrically cut it.

In the Fourth Book.

1. To inſcribe a Line in a Circle.
2. To inſcribe a Triangle in a Circle.
3. To circumſcribe a Triangle about a Circle.
4. To inſcribe a Circle in a Triangle.
5. To circumſcribe a Circle about a Triangle.
6, 7. To inſcribe and circumſcribe a Square in or about a Circle.
8, 9. To inſcribe and circumſcribe a Circle about a Square.
10. To make an Iſoſceles-Triangle, whoſe Angle at the Baſe is double the Angle at the Vertex.
11. To inſcribe a regular Pentagon, or any other Polygon in a Circle: On a given Line to deſcribe a Pentagon.
12. To circumſcribe a regular Pentagon, or any Polygon.
13, 14. To inſcribe in, or circumſcribe a Circle about a regular Polygon.
15. On a Line given to deſcribe an Hexagon, or to inſcribe it, or an Equilateral Triangle in a Circle.
16. To inſcribe a regular Quindecagon, or innumerable regular Figures.
    On a given Line to deſcribe any regular Figure.
    What Number of regular Figures will fill a Space, *i. e*, whoſe Angles about one Point, juſt make 360 Degrees.

In the Sixth Book.

1. To divide a Trapezium.
6. To make Similar Triangles.
9. To divide a Line in a given Proportion.
10. To divide a Line as another is, or into any Number of Parts equally.
11. To find a third Proportional to two given Lines, alſo the Sum of infinite Proportionals.
12. To find a fourth Proportional to three Lines given.
13. To find a mean Proportional, or two Means betwixt two given Lines divers Ways, *L.* 12. 18.
14. To demonſtrate the Inverſe Rule of Proportion.
16. To demonſtrate the Direct Rule of Proportion.
18. On a right Line to deſcribe a Polygon like and alike ſituate, to a given one. Hence the drawing and reducing Maps. See 20, 21.
19. Similar Figures are in a duplicate Ratio of like Sides. See 20. 23. *L.* 12. 2:
25. To make a Polygon equal to a given one, and like to another given one.
31. To add or ſubtract right-lin'd Figures, and to ſquare the Lunets of *Hippocrates.*

In the Eleventh Book.

11. To draw a Perpendicular to a Plane.
12. To erect a Perpendicular on a Plane.
21. A Demonſtration of the Five regular Bodies.
33. Like Bodies are to each other, as the Cubes of their homologous Sides, *L.* 12. 9.

In the Twelfth Book.

18. To encreaſe or diminiſh Solids.

By this Synopſis, and the following Appendix, it will appear what a large *Apparatus* towards a Syſtem of *Practical Geometry* is already Extant in our Language, for want of which there has been theſe many Years paſt, no ſmall Complaint: Eſpecially if to this be added *Le Clerc*'s Geometry, *Hawney*'s Complete Meaſurer, and *Langley*' Practical Geometry; to the firſt of which, I am not a little in debt for ſeveral of the following Problems.

Problem

Problem I.

TO divide a right Line (AB) into any Num- of equal Parts, suppose 8.

For this purpose have in readiness a Number of equi-distant Parallels drawn and numbred from 0 to 12 or 20, with the Distance of A B given. Set one Foot of the Compasses on the Parallel mark'd 0.0, and turn the other till it touch the eighth Parallel; unto which two Feet apply a Ruler, and draw the Line A B, which the Parallels will equally divide into eight Parts. *W. W. D.*

This is so plain and easy, as not to need any Figure.

Problem II.

TO find the Area of a plain Triangle, having the three Sides, without a Perpendicular.

Add the three Sides, and take half that Sum; then subtract each Side severally from that half Sum: Multiply that half Sum and the three Differences continually, and out of the last Product extract the square Root, which is the Area of the given Triangle.

Or,

Having the Diameter of the inscribed Circle, multiply it by a fourth of the Sum of the three given Sides for the Area sought.

Problem III.

HAVING the Area of any Triangle, to find the Diameter of the inscribed Circle.

Divide the said Area by a fourth of the Sum of the three Sides; the Quotient is the Diameter of the inscribed Circle.

Problem IV.

TO reduce any Figure into a Square; or to make a Square equal to a Parallelogram, Triangle, Trapezium, Polygon or Circle, &c.

By the Rules of Mensuration, cast up their Area's, the square Root whereof is the Side of the Square sought.

Or

Or Geometrically thus:

Consider what two Things multiplied together produce their Area: Between those two Numbers, or Sides, find a Geometrical Mean by *Lib.* 6. 13 *Euclid.* This Mean is the Side of a Square equal. *W. W. D.*

N. B. The two Things which multiplied, produce their Area in a

Parallelogram rectangled, Rhombus, Rhomboides,	are	The Length and Perpendicular Breadth.
Triangle,	are	Base and half Perpendicular.
Trapezium,	are	Diagonal and half of both Perpendiculars.
Circle,	are	Half Diameter and half Periphery.
Polygon,	are	Half Perimeter and the Perpendicular from the Center.
Sector,	are	Half the Arch and half the Diameter.
Segment,	are	Chord, and two Thirds of the V. Sine.

N. B. If it be an irregular Figure, reduce it into Triangles, or Trapezia's, by drawing Lines within from Corner to Corner; then reduce the said Triangles, *&c.* into Squares, which Squares add by 47. 1. or 14. 2. *Euclid.*

Problem

Problem V.

TO find a strait Line (EI) nearly equal to the the Arch of a Quadrant (B C.)

Fig. 6. Bisect the Quadrant BD in F, and draw BF; to BF erect the Perpendicular A B on B; then set A C from C to I, so will E I be nearly equal to the Arch of the Quadrant B D or BC.

Or,

Bisect B D in F, and B C in G, then with the Distance KF, on the Center K, draw the Arch H F G I, so will the Line H I be nearly equal to the Semi-circumference, and E I to the Quadrant, as before.

N. B. Archimedes demonstrated, that the Circumference is to the Diameter less than as 22 : 7, and greater than $21\frac{70}{71}$ to 7; within which strict Limits, many Years ago, it was found in whole Numbers to be as 9 : 10 : : Chord : Arch of a Quadrant.

Suppose then that the Diameter=7, its half 3.5, the Square of which is 12.25; which doubled, is 24.5, whose square Root is 4.949, the Chord of the Quadrant; then say, as 9 : 10 : : 4 × 4.949 : 21.995, the Circumference, which is greater than $21\frac{70}{71}$, or 21.985. Again, 4.95 is something greater than the Chord, which × 40, and divide by 9, as above, gives 22; so that by this Proportion, as 9 : 10, a like Answer with *Archimedes* is also discover'd, *i. e.* less than as 22 : 7, and greater than as $21\frac{70}{71}$ to 7.

Con-

Concerning Squaring the Circle, I offer ſome few Eaſy and Practical Obſervations following.

1. A Circle is equal to a right-angled Triangle, whoſe Baſe is the Circumference of the Circle and its Perpendicular the Radius of it.

2. Every Polygon circumſcribed is greater, and every Polygon inſcribed is leſs than the Circle.

3. The Compaſs of a Polygon circumſcribed is greater, and inſcribed is leſs than the Circumference of a Circle.

4. This right-angled Triangle aforeſaid, will be leſs than any Polygon circumſcribed, and greater than any inſcribed: Becauſe the Circumference of this Circle (which is the Baſe of this Triangle) is greater than the Compaſs of any inſcribed, and leſs than the Compaſs of any circumſcribed Polygon; therefore it will be equal to the Circle.

Becauſe every imaginable inſcribed Figure, which is leſs than the Circle, is alſo leſs than the Triangle; and every circumſcribed Figure, greater than the Circle, is alſo greater than the ſaid Triangle likewiſe; therefore the ſaid Triangle is equal to the Circle: But actually to find a right Line exactly equal to the Circumference, is not yet diſcover'd Geometrically.

Howbeit, I ſhall offer an eaſy Approximation to the young Student, whom I deſire to account the Circle a Polygon of 1000.10000 or 43200 Sides, *i. e.* the half Minutes in 360, ſo ſmall a Part of a Circle will inſenſibly approach to, or become very nearly a ſtrait Line; nay, the Product of the Sine and Tangent of one Minute multiply'd by the ſaid 43200, will agree in the laſt 7 Figures, *i. e.* 6283185, whereby we obtain the Circumference of the Circle, whoſe Radius is 1000000, to the like Number of Figures exactly, and may to as many more Places, if we add the two Products together, and take the half for the Circle's Circumference, which will be leſs than the *Tangent-Product*, or circumſcribed Polygon, and greater than the *Sine Product*, or inſcribed Polygon.

S For,

For, according to the preceding third Propoſition out of *Archimedes*, the Circuits or Polygons circumſcribed about and inſcribed in a Circle, do at laſt end in the Circumference of the Circle; in like manner the Polygons themſelves do at laſt end in a Circle.

To this I ſubjoin certain Practical Remarks on Regular Polygons, their Angles and Sides.

REgular Polygons may be delineated ſeveral Ways; Firſt, By the Angle at the Center; Secondly, By the Angle at the Figure; Thirdly, By the Angle of the Triangle at the Baſe.

To find Each.

Divide 360 by the Number of Sides, the Quotient is the Angle at the Center, whoſe Complement to 180 is the Angle at the Figure, and half the ſaid Complement is the Angle at the Triangle: By this one may make a Table of Angles for as many Polygons as he pleaſes.

Polygons of odd Number of Sides are inſcribed in Circles by the help of *Iſoſceles* Triangles, whoſe Angles at the Baſe are Multiples of that at the Top.

If the Angle at the Baſe be	Double,	the Angle at the *Vertex*, the Baſe will be the Side of a	Pentagon.
	Triple,		Heptagon.
	Quadruple,		Enneagon.
	Quintuple,		Endecagon.

The Number of Degrees of the Angle at the *Vertex*, is found by dividing 180 by the Number of *Sides* in the *Polygon* to be inſcribed, the Quotient gives the ſaid Angle; which doubled, tripled, &c. will give the Angle at the Baſe.

Seeing Polygons inſcribed are the Chords of the Angles at the Center, which Chords are always double the Sine of half the Angle at the Center. Therefore in the Table of Natural Sines, hunt out the Sine of half the Angle at the Center, which doubled, is the exact Side of the Polygon, in ſuch Parts as the Radius contains 10000000, &c.

A Table

A Table for the inscribing and describing Polygons.

Number of Sides.	Quantity of the Side.	Angles at the Center.	Angles at the Figure.	Angles at the Triangle.
3	17320508	120	60	30
4	14142135	90	90	45
5	11755705	72	108	54
6	10000000	60	120	60
7	8677674	$51\frac{3}{7}$	$128\frac{4}{7}$	$64\frac{2}{7}$
8	7653668	45	135	$67\frac{1}{2}$
9	6840402	40	140	70
10	6180339	36	144	72
11	5634651	$32\frac{8}{11}$	$147\frac{3}{11}$	$73\frac{7}{11}$
12	5176380	30	150	75

The Use.

To inſcribe a Heptagon in a Circle, whoſe Radius is 500; ſay as *Tabular Radius* 1000 : *Tabular Heptagon* 867 : : ſo is the Given Radius : Side Heptagon ſought 433.

To deſcribe a Heptagon on a given Line, find the Radius proper; ſay as the *Tabular Heptagon* 867 : *Tabular Radius* 1000 : : ſo is the Given Side 433 : to the proper Radius ſought 500.

Problem VI.

TO make a Circle (ADG) nearly equal to a given Square (AECF.)

Draw the Diagonal AC, which divide into 10 equal Parts; 8 of them is nearly equal to the Diameter of the Circle ſought: Wherefore in the middle of the Diagonal AC, and with the Radius of 4 of thoſe equal Parts draw the Circle ADG. *W. W. D.* *Fig.* 8.

Or thus more briefly;

Biſect EC in D, and draw AD, the Diameter of the Circle ſought.

Problem VII.

TO make an Iſoſceles Triangle (BCE) equal to a given Square (ABCD.)

Fig. 8. Extend the Side C D to E, making D E=C D, and draw B E, ſo will B C E be equal to A B C D. *W. W. D.*

To make an Equilateral Triangle equal to a Square, ſome adviſe to make the Side of the Triangle one and a half of the Side of the Square; but it is ſomewhat too little.

Problem VIII.

WIthin a given Triangle (KLM) to inſcribe a Square (NOPQ.)

Fig. 9. On M erect the Perpendicular M H, equal to L M; let fall the Perpendicular K G, then draw G H, cutting K M in N; through N draw N O, parallel to the Baſe, L M, and N Q, and O P parallel to K G, ſo is N O P Q the greateſt inſcribed Square. *W. W. D.*

Problem IX.

WIthin the Square (RSTV) to inſcribe an Equilateral Triangle (RZY.)

Fig. 10 Draw the two Diagonals S T and R V; on V, as a Center, with the Extent V X, draw the Arch Z X Y; then draw the Lines R Z, Z Y, R Y, which form the Triangle R Z Y. *W. W. D.*

Problem X.

ABOUT an Equilateral Triangle (ABC) to deſcribe a Square (AFDG.)

Fig. 11. Biſect B C in H, and draw A H D; make H D equal to HC, and biſect A D in E; through E draw FEG at right Angles, making F E and E G each=E D, then draw the Lines A F, F D, D G and A G, forming the Square. *W. W. D.*

Problem XI.

ABOUT a Square (LMNO) to draw a Triangle (PQR) whoſe Angles ſhall be equal to the ſeveral Angles of a Triangle given (STV.)

Extend

Extend N O to P Q, then make the Angle R L M=T and R M L=V, then draw the Sides R L P and R M Q, so is R P Q the Triangle. *W. W. D.* Fig. 12.

Problem XII.

TO inscribe an Equilateral Triangle (AFG) in a given Pentagon ABCDE.)

Find its Center H, and draw the Circle about it; then inscribe a Triangle in that Circle, which will also be the Triangle in the Pentagon. *W. W. D.* Fig. 13.

Or,

With the Radius A H, and Center A, draw the Arch H M, which bisect in K, and draw A K F, the Side of the Triangle sought.

Problem XIII.

TO inscribe the Square (MNOP) in the Pentagon (ABCDE.)

Extend the Perpendicular A Q and B C till they cut each other in F; make the Perpendicular F I and A H equal to half A F; then draw B H and B I, cutting A Q in L and K, so will L K be the Side of the Square sought. Fig. 14.

Problem XIV.

OF four given Lines (AB=6, BC=9, CD=8, AD=18) the greatest being less than the Sum of the rest, to make a Quadrangle which may be inscribed in a Circle.

In every Quadrangle inscribed in a Circle, the Sum of the Rectangles made of the Sides, containing opposite Angles, have the same Proportion to each other as the Diagonals, *i. e.* as, Fig. 15.

$$\overset{72}{DCB}+\overset{108}{DAB}\ \text{to}\ \overset{54}{ABC}+\overset{144}{CDA},\ \text{so is}\ CA\ \text{to}\ DB$$
$$180 \qquad\qquad 198 \qquad\qquad 13+ \quad 15+$$

 Also,

Also,

$$\overset{144}{\underset{198}{ADC+ABC}} : \overset{72\quad 108}{\underset{180}{BCD+DAB}} :: \overset{162\quad 48}{\underset{210}{AD\times BC+AB\times CD}} : \underset{190\frac{10}{11}}{AC\square}$$

The Root is 13 +

Again,

$$\overset{72\quad 108}{\underset{180}{BCD+DAB}} : \overset{144\quad 54}{\underset{198}{ADC+ABC}} :: \overset{162\quad 48}{\underset{210}{AD\times BC+AB\times CD}} : \underset{231}{BD\square}$$

The Root is 15 +

From the last of these Proportions find the Diagonal BD; with which, and the other two Sides, either B C and C D, or A B and A D, form a Triangle, and about it draw the Circle A B C D, and in it insert the other two Sides of the Quadrangle. *W. W. D.*

Problem XV.

To inscribe an Heptagon, Nonagon, or Undecagon, &c.

Fig. 16. Having obtained the Side of a Polygon, next bigger and next less, in the same Circle, extend the Diameter A B to D, and from C extend the said Sides of the Polygon next above or under to D and E, which Distance bisect in F, and draw C F, the Side of the Polygon sought lies on that Line CF, between C and the Circumference.

The Side of a Septagon or Heptagon, is nearly equal to the Perpendicular of an Equilateral Triangle, whose Side is Radius, or an Hexagon; so the Side of a Nonagon is nearly the Perpendicular of an Equilateral Triangle, whose Side is an Octagon.

Problem XVI.

To describe a regular Octagon on a given Side (A B.)

Fig. 17. On the middle of A B, erect the Perpendicular C E; on the Point C and Distance AC, describe a Semi-circle ADB; on

on the Point D, and with the Diſtance D A, draw A E B; ſo is E the Center, and A E the Radius of that Circle, which contains the Oʃtagon, whoſe Side is A B. *W. W. D.*

Problem XVII.

TO deſcribe a regular Nonagon on a given Line (A B.)

Ereʃt the Perpendicular F C on the middle of A B; on B, with the Diſtance A B, draw the Arch A D, which biſeʃt in E; on the Point D, with D E, draw the Arch E F; ſo is F the Center, and A F the Radius of that Circle which contains the Nonagon, whoſe Side is A B. *W. W. D.* *Fig.* 18.

Problem XVIII.

TO inſcribe a regular Nonagon in a given Circle.

On B, with the Radius A B, draw D A C and D C, which extend to F; make E F = A B; on E and F draw E G and F G, then draw A G, which cuts the Circle in H; ſo is D H the Side of the Nonagon ſought. *W. W. D.* *Fig.* 19.

H B is the Side of a regular Polygon of 18 Sides, and $\frac{2}{3}$ of the Radius is the Side of a regular Nonagon.

Problem XIX.

TO inſcribe a regular Undecagon in a Circle.

Divide A B in two at C; on A and C, with the Diſtance A C; draw the Arch C D I and A D; on the Center I and Diſtance I D, draw the Arch D O; ſo is C O the Side of the Undecagon ſought. *Fig.* 20.

Some ſay $\frac{9}{32}$ of the Diameter is the Side of an Undecagon inſcribed, or $\frac{3}{4}$ Diameter more, $\frac{1}{8}$ of the ſaid $\frac{1}{4}$.

Or,

Quarter a Circle and inſcribe an Equilateral Triangle, that Part of the Side which lies betwixt the Diameter and the Angle of the Baſe is the Side of a regular Undecagon.

Problem XX.

TO describe a regular Dodecagon on a given Side (AB.)

Fig. 21. On the middle of A B, erect the Perpendicular C D; on A and B, with the Distance A B, draw the Arches A E and B E; on E, with the Distance A E, draw the Arch A D: so is D the Center of the Dodecagon sought: For A E B is the Angle of the Hexagon, and A D B is its half, the Angle of a Dodecagon.

Problem XXI.

ON a given Line (A B) to describe any Polygon from 6 to 12 Sides.

Fig. 22. Bisect A B in O; erect on O, the Perpendicular O I; on B, with the Distance AB, describe the Arch AC, which divide into 6 equal Parts from C, and from the Distance of each of those equal Parts draw the Arches D M, E N, F P, G Q, H R and A I; so is

C	Center of a	Hexagon,	and	C A	the Radius.
D		Heptagon,		D A	
E		Octagon,		E A	
F		Nonagon,		F A	
G		Decagon,		G A	
H		Endecagon,		H A	
I		Dodecagon,		I A	

Problem XXII.

TO cut two given Lines (AB and AC) into four, so that all four shall be continually proportional, CF : DF : : ED : BE.)

Fig. 23. Set A B and A C at right Angles, and draw B C; bisect A B, and draw the Semi-circle B D O through D; draw D F parallel to B O, and D E parallel to C O; so will it be,

B E : E D : : E D : D F
E D : D F : : D F : F C

Problem

Problem XXIII.

FROM a ſtrait Line given (I) to cut off a Part (G D) which ſhall be a Mean betwixt the other Part and another Line given (H.)

Make the Line C D = Lines H and I given, on E, where the two Lines meet, erect the Perpendicular E F, and draw the Semi-circle on the Diameter C D. Biſect C E = H in B; on B, with the Radius B F, draw the Arch F G; ſo will D G : G E : E C ÷: *Fig.* 24.

Problem XXIV.

GIVEN the Sum of the Extremes (A B) and the Mean (B D) to find the Extremes (A F and F B) ſeverally.

On A B erect a Semi-circle, and on B erect a Perpendicular equal to B D; through D, parallel to A B, draw E D and E F parallel to D B; ſo is A F one and F D the other Extreme. *Q. E. I.* *Fig.* 25.

Problem XXV.

GIVEN the Difference of the Extremes (A B) and the Mean (B C) to find the Extremes (B E and B F) ſeverally.

On A B, the Difference, erect the Perpendicular B C = Mean. Biſect A B in D; on D, with the Diſtance C D, draw the Semi-circle, whoſe Diameter F E contains the two Extremes ſought, *i. e.* B E and B F. *Fig.* 26.

Problem XXVI.

THE Exceſs of the Diagonal above the Side of a Square being given (A B) to find the Side (A D).

Arith.

Arithmetically thus:

Fig. 27. To the Excess given A B, and the square Root of double the Square of the Excess for the Side A D sought.

Geometrically:

On B erect the Perpendicular B C = A B, and draw ACD; on C, with the Distance CB, draw the Arch B D; so will A D be the Side of the Square, and A E the Diagonal, which exceeds A D by A B.

Problem XXVII.

GIVEN the Sum of the Side and Diagonal (AB) to find them separately (G A and G B.)

Fig. 28. On AB draw the Semi-circle ACB; erect C D perpendicular. Bisect A D in E; on E, and the Distance E C, draw C F; the Chord C F is the Diagonal sought, which set from B to G; so will G A be the Side sought.

Problem XXVIII.

GIVEN the Area of a rectangled Parallelogram, (36) and the Proportion of the Sides, as 4 to 1, to find the Sides.

Say 4 : 1 : : 36 : 9, whose square Root is 3, the Breadth; as 1 : 4 : : 36 : 144, whose square Root is 12, the Length.

Problem XXIX.

GIVEN the Difference of the Sides and Sum of the Sides of a right-angled Parallelogram, to find the Sides.

To the half Sum add the half Difference for the Length; from the half Sum take the half Difference for the Breadth.

Problem

Problem XXX.

GIVEN the Difference of the Sides, and of their Squares, to find the Sides of a rectangled Parallelogram.

Divide the Difference of the Squares by the Difference of the Sides; the Quotient is the Sum.

Or,

Divide the Difference of the Squares by the Sum of the Sides, the Quotient is the Difference of their Sides, by 6. 2 *Euclid.*

Problem XXXI.

IN any Triangle, the Sum of every two Sides given, to find them severally.

From half the Sum of all the given Numbers, ſubtract each, and the Remainders are the Sides required, *i. e.* each particularly of the Letter wanting.

Problem XXXII.

THE Sides of a Trapezium, (ABCD) and one Diagonal (AC) given, to find the other (BD.)

□ A C + B C — A B ; divide half the Remainder by AC, the Quotient is C E. Fig. 29.

□ A C + A D — C D ; divide half the Remainder by A C, the Quotient is A F, by 47. 1. Find B E and D F, add C E and A F, and ſubduct the Sum from A C, the Remainder E F = D G, then □ D G + □ B G = □ B D. *Q. E. I.*

Problem

Problem XXXIII.

IN a right-angled Triangle (ABC,) right-angled at B, is given AC=13, and the Sum of AB and BC=17, to find AB and BC apart.

Fig. 30. Make the Square E D=17 for the Side, and 289 will be the Area; from whence take the Square of A C=169, the Remainder 120=2 Rectangles E B and B D, each of which is a Mean betwixt the Square A B and B C, and double the Triangle A B C

Wherefore take half 17=8.5, whose Square is 72.25: from which take 60, and there rest 12.25, whose square Root is 3.5; which taken from 8.5, leaves 5 for C B, and added to 8.5, make 12 for A B.

Like to this is the following. " In a right-angled Triangle A B C, right-angled at B, is given B C=40, and the Sum of A B and A C together 150, to find them severally.

Square the Perpendicular, and divide that Sum 1600 by double the Sum of A B and B C=300, the Quotient is $5\frac{1}{3}$, which added to the half Sum of A B and B C=75, gives $80\frac{1}{3}$ for the Hypothenuse A C; and taken from it, gives $69\frac{2}{3}$ for A B.

Or,

From the Square of the Sum of A B and A C 150=22500 be subtracted the Square of B C 40=1600 and divide the Remainder =20900 by double the Sum of A B and B C=300, the Quotient is $69\frac{2}{3}$ for A B, &c.

Problem XXXIV.

IN the Rectangle (ABCD) are given BE=16 and EF=2. Quære the Area,

Fig. 31. Seeing the Angles are right, 'tis as FE : ED :: ED : EA; and again, as E D : E A :: E A : E B. And seeing E B is the third Number, and E D and E A are two Means proportional betwixt 2 and 16, reduce each into their least Proportion, which is 1 and 8, and extract the Cube Root of 8 is 2, which doubled (because 2.16 were halved before) give 4 for E D; then is E A 8, and consequently the Area 160.

Problem XXXV.

IN the Triangle (ABC) are the three Perpendiculars ($AD=56$, $BF=60$, and $EC=64\frac{8}{13}$) given, to find the three Sides.

In the Triangle, whose Sides are 13, 14, 15, the Perpendicular is found to be 12; wherefore say, as Fig. 32.

$$12 : 14 :: \begin{cases} 56 & : 65\frac{1}{3} \quad AB \\ 60 & : 70 \quad AC \\ 64\frac{8}{13} & : 75\frac{5}{13} \quad BC. \end{cases}$$

Problem XXXVI.

IN an Isosceles Triangle (ABC) are two Circles touching one another; and the Sides of the Triangle, the Diameter of one is 12 and the other 8. Quære the Sides of the Triangle.

E H is the Difference of their Radii $=2$, and E F the Sum of the Radii $=10$,; then say, if 2 give 10, what will $EG=6$ give? Answer, 30 for A E; to which add $ED=6$, and it makes 36 for A D, the Perpendicular; then $\square AE-\square EG=\square AG$; then, as $AG:AE::AD:AC=AB$; then $\square AC-\square AD=CD$, which doubled, is B C the Base. *Q. E. I.* Fig. 33.

Problem XXXVII.

THE Hypothenuse $AB=48$, and the Area of a right-angled Triangle $=384$, given to find the Sides AC and CB.

On $AB=48$, the Hypothenuse, make a Semi-circle, on whose Center D, erect the Perpendicular D G; then double the Area given 768, which divide by the Hypothenuse $=48$, the Quotient is $DE=16$ of the Isosceles Triangle A E B; through E, parallel to A B, draw E C, then A C and B C are the Sides sought. *Q. E. I.* Fig. 34.

$\square DC-CF\square=DF\square$, then $AD+DF=AF$, and $\square AF+FC\square=AC\square$, then $AB-AF=FB$, and $\square CF+FB\square=BC\square$.

Problem

Problem XXXVIII.

THE Diameter of a given Circle $=10$, *in which make the greatest Rectangle possible, whose Length shall be double its Breadth. Quære the Sides.*

Fig. 35. Divide the Square of the Diameter by 5, the Quotient is 20, whose square Root is the Breadth: Deduct 20 out of 100, remains 80, whose square Root is the Length. *Q. E. I.*

Problem XXXIX.

OHE Diameter of a Circle $=8$, *in it to inscribe on the Diameter an Equilateral Triangle, whose Vertex shall touch the Circumference. Quære its Side.*

Fig. 36. Cross the Circle with the two Diameters, A B, E F; in it inscribe an Equilateral Triangle A C D, whose Vertex shall be A, so will the Diameter E F cut off the Equilateral A H K. *W. W. D.*

To find the Side.

$$\square AG + \tfrac{1}{3}\square AG = \square AH.$$

Problem XL.

IN the Circle (AFBGCD) having the Square (ABCD) inscribed, is given each Segment's Area $=224$ *(ABF) contain'd betwixt the Side of the Square and the Circumference, to find the Diameter.*

Fig. 37. Suppose the Diameter $=1$, the Area will be $\frac{11}{14}$, whereof $\frac{1}{4}$ will be $\frac{11}{56}=$ A E B F, the Area of the Triangle A E B $=\frac{1}{8}$, which

which ſubtracted from $\frac{11}{56}$ leaves $\frac{1}{14}$ for A B F: then ſay, as $\frac{1}{14}$: □ 1 : : 224 Quantity given : □ Diameter = 3136, whoſe ſquare Root is 56 = Diameter A C, ſuppoſing the Proportion of the Diameter to the Circumference, as 7 to 22.

Problem XLI.

IN a Circle, whoſe Diameter (*BD*) = 65 *is a Triangle inſcribed* (*A B C*) *the Side* (*A B*) = 52 *and* (*B C*) = 56. *Quære the Side* (*A C.*)

□B D — □A B = □A D and □B D — □B C = □C D, *Fig.* 38. then multiply A D by B C, alſo A B by D C; theſe two Products = 3900 divide by B D = 65, the Quotient 60 = A C. *Q. E. I.*

Problem XLII.

IN the Circle (*A C E B*) *is inſcribed a Triangle* (*A B C*) *A B* = 13, *B C* = 14, *A C* = 15, *and in that Triangle is inſcribed a Circle. Quære the two Diameters of both Circles.*

Find the Perpendicular A F = 12, then becauſe the Triangles A C E and A B F are ſimilar, ſay A F : A B : : A C : A E, the Diameter of the circumſcribing Circle = $16\frac{1}{4}$. *Fig.* 39. Then ſeek the Area of the Triangle = 84, which doubled is 168; then add all the three Sides, they make 42, by which divide 168, the Quotient is 4 = G D, half the Diameter of the inſcribed Circle.

Then

The Proportion *of the* Sphere, *and of the Five* Regular Bodies, *inscribed therein, from* Peter Herigon, *Cursus Math.* Vol. I. p. 779.

DIAMETER of the Sphere 2. Circumference of the great Circle 6.28318, Area thereof 3.14159, Area of the Sphere 12.56637, Solidity of it 4.18859.

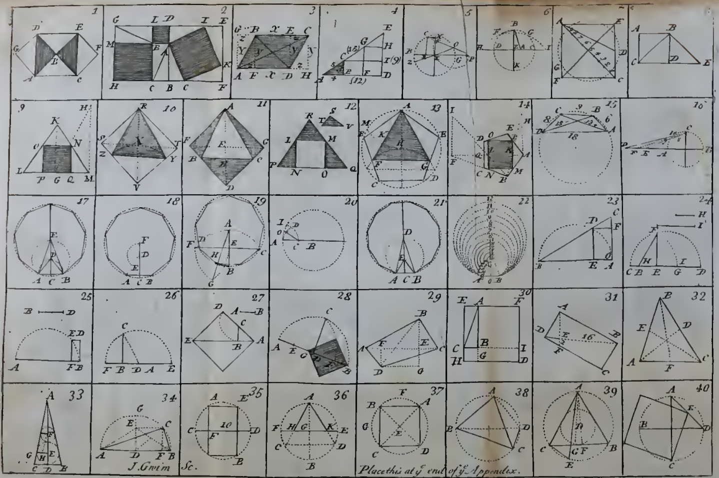
	Tetraedram.	*Hexaedrum.*	*Octaedrum.*	*Dodecahedrum.*	*Icosihedrum.*
Side,	1.6229	1.1547	1.4142	0.7136	1.0514
Surface,	4.6188	8.	6.9282	10.5146	9.5745
Solidity,	0.1513	1.5396	1.3333	2.7851	2.5361

Problem XLIII.

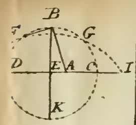
TO find a Cube nearly equal to a given Sphere (ABCD.)

Fig. 40. Quarter the Circle with AC and BD, and draw AD, which bisect in E; then draw CE, the Side of a Cube, nearly equal to the Sphere ABCD. *W. W. D.*

FINIS.


J. Gwin Sc.
Place this at yᵉ end of yᵉ Appendix.

6



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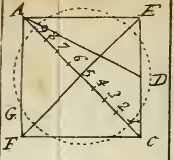

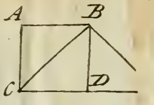

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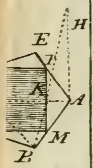

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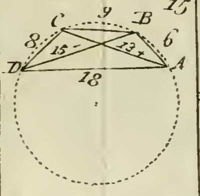

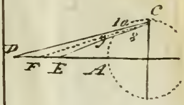

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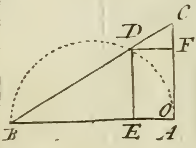

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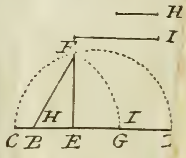

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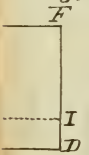

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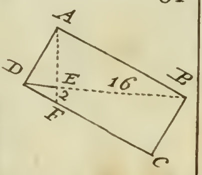

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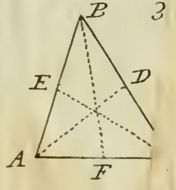

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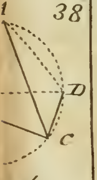

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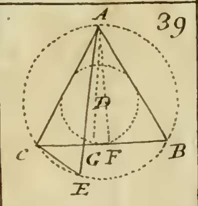

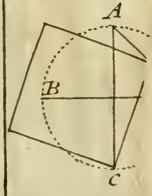

ndix.

S

6

5 19

6 11

11

66

5

330

35

$6ax + a - b = 2bx - 2a$

$6ax - 2bx = 3a + b$

$\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 1$

[illegible]

$3x + 2x - \frac{6x}{4} = 3$

$12x + 8x - 6x = 12$

$14x = 12$

$x = \frac{12}{14} = \frac{6}{7}$